Convergence rate analysis of time discretization scheme for confined Lagrangian processes

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Abstract

In this paper, we propose and analyze the convergence of a time-discretization scheme for the motion of a particle when its instantaneous velocity is drifted by the known velocity of the carrying flow, and when the motion is taking into account the collision event with a boundary wall. We propose a symetrized version of the Euler scheme and prove a convergence of order one for the weak error. The regularity analysis of the associated Kolmogorov PDE is obtained by mixed variational and stochastic flow techniques for PDE problem with specular condition.

Key words: Lagrangian stochastic model; time-dircretisation scheme; weak error;

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1 Introduction

Many industrial production processes involve suspensions of colloidal particles in fluids so there is a strong interest to better understand the underlying physics. Among the ways that can help to achieve this goal, numerical experiments combining the simulation of the flow and the simulation of the particles carried by the flow is a possible solution. Propositions of model-motion of colloidal particles are already well-known, assuming that they can be modeled by small spheres and that the description of the model motions of their gravity centers is a significant approximation when one want to asses some characteristic behavior through collision kernel modeling.

In this paper, we propose and analyze the convergence of a time-discretization scheme for the motion of a particle when the instantaneous velocity of the particle is drifted by the known velocity of the carrying flow, and when the motion is taking into account the collision event with a boundary wall.

More precisely, since we want to work in a context where we can specify the mathematical well-posedness of the problem and regularity for the solutions of associated PDEs, some simplifications are considered. We assume that the collision is perfectly elastic and that the particles follow a kinetic model, by modeling the position and velocity of each particle. It is on the velocity that we introduce a drift term to model the influence of the fluid on the particles. Furthermore, we will only consider a particle that collides against a wall located at the boundary of the upper-half plane $\mathbb{R}^{d-1} \times [0, +\infty)$. In this case the confined linear Langevin process is written as:

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t U_s\, ds, \\ U_t = U_0 + \displaystyle\int_0^t b(X_s, U_s)\, ds + \sigma W_t + K_t, \\ K_t = -\displaystyle\sum_{0<s\le t} 2\,(U_{s^-} \cdot n_{\mathcal{D}}(X_s))\, n_{\mathcal{D}}(X_s) \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}, \end{cases} \tag{1.1}$$

where $(X_t)_{t\ge0}$ represents the position while $(U_t)_{t\ge0}$ represents the velocity, $\mathcal{D} := \mathbb{R}^{d-1} \times (0, +\infty)$ is the open set corresponding to the interior of the confining domain, $n_{\mathcal{D}}$ is the outward normal at the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ ($\partial\mathcal{D} = \mathbb{R}^{d-1} \times \{0\}$) and σ is a positive constant. Here the drift b models the drag force implied by the known mean velocity of the flow carrying the particle. The term $(K_t)_{t\ge0}$ represents the perfectly elastic collision with the hyperplane $\partial\mathcal{D}$.

Although simple -known as specular reflection against a fixed wall- this model contains enough characteristics of the context stated in the first paragraph to be pertinent on a framework of numerical analysis. In [5], Bossy and Jabir prove the existence of weak solution and pathwise uniqueness when $\mathcal{D} = \mathbb{R}^{d-1} \times (0, +\infty)$. In [6], the authors extend the well-posedness result to smooth bounded domains $\mathcal{D}$. In the case of hyperplane $\mathcal{D} = \mathbb{R}^{d-1} \times (0, +\infty)$, the construction proceeds as follows (see [5] for the details). If we consider a $\mathbb{R}^d$-valued bounded measurable drift $\tilde{b}$ on $\mathcal{D} \times \mathbb{R}^d$, from the unique weak $\mathbb{R}^{2d}$-valued solution of

$$\begin{cases} Y_t = X_0 + \displaystyle\int_0^t V_s ds, \\ V_t = U_0 + \displaystyle\int_0^t \tilde{b}(Y_s, V_s) ds + \widetilde{W}_t,\ \forall\, t \in [0, T], \end{cases} \tag{1.2}$$

with $\tilde{b}$ defined by

$$\tilde{b}\colon (y, v) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \left(b', \operatorname{sign}(y^{(d)}) b^{(d)}\right)\left((y', |y^{(d)}|), (v', \operatorname{sign}(y^{(d)}) v^{(d)})\right) \tag{1.3}$$

then

$$((Y_t', |Y_t^{(d)}|), (V_t^{'}, \mathrm{sign}(Y_{t^+}^{(d)})V_t^{(d)}); t \in [0,T])$$

is the weak solution in $\mathcal{D} \times \mathbb{R}^d$ to the SDE (1.1).

A short discussion on confined SDEs and associated results

There are different types of confined models that can be considered. In a deterministic setting, [23] present some results when $\sigma = 0$ in (1.1) while allowing for oblique reflections. The authors show that the system admits a solution such that the position process is Lipschitz continuous in time, and the velocity process is of bounded variation. This solution is obtained as a certain weak limit in a Sobolev space of solutions to a penalized equation.

The most obvious stochastic model would be a diffusion that is reflected at the boundary, in the sens of a solution to a Skorohod problem as in [20]. The reflection term K is then given through a local time. In term of discretisation scheme, [4], propose a symmetrized scheme, and prove that the associated weak error has a rate of convergence of order one.

In [11], the author presents a model with a particle that exhibits piecewise deterministic movement. The velocity process changes randomly at exponential times to mimic the collision events. The particles are confined in domain by specular reflections at the boundary. It is shown that such a system is well defined and by increasing the change rate for the velocity, in the limit, one obtains an oblique reflected diffusion.

We emphasize the fact that, when modelling the position of the particle by a reflected Brownian process, the hitting times of the boundary form almost surely a set of times with no isolated points. This means that it is impossible to count the number of collisions with the boundary. Those models are not suitable in numerical approach when one might to determine a collision kernel with the help of the effective collision rate. Such inconvenient disappears by considering models for the particle collisions of Lagrangian type, where the position process is the integral of a diffusion. As shown in [21], situation of accumulation of collisions can be avoided for Lagrangian models in the case of a upper half plane under the hypothesis that $(X_0, U_0) \neq (0,0)$.

We also mention that the case of absorbing boundary have been studied in [2] and in [16], [17] who prove the existence of a reflecting Langevin process with an absorbing boundary.

Finaly, in [12] and in [25], it have been shown that using a certain type of scaling and limit in the drift and diffusion parameters in (1.1), it is possible to pass from a Langevin model with specular reflection (1.1) to a reflected diffusion model for the position process.

Discretization scheme for the confined SDE (1.1)

Without any loss to the generality, we present a discretisation scheme in case of the dimension $d = 1$. The scheme can be easily generalized to higher dimensions by combining the discretization of the first $d-1$ components of the process $(X_t, U_t)_{t\geq 0}$, solution of (1.1), using standard discretization scheme in $\mathbb{R}^{d-1}$, and the confined scheme presented in this section for the dth component.

As previously mentioned, in [5] the authors construct a weak solution to the equation (1.1) when the reflection border is a hyperplane. The position process of this weak solution is written as the absolute value of an unconfined Langevin process. The following scheme borrows the main ideas of this transformation by symmetry.

The confined process is discretized on an a regular mesh $0 = t_0 < t_1 < \ldots < t_n = T$ of the interval $[0,T]$. $\Delta t = t_{i+1} - t_i$ is the time increment. We define the discretized process $(\bar{X}_t, \bar{U}_t)_{0\leq t\leq T}$ with an iterative procedure. Knowing (X_{t_i}, U_{t_i}) we construct $(X_{t_{i+1}}, U_{t_{i+1}})$ as follows:

• **Discretization of the position process.** We denote by $(\bar{Y}_t)_{0\leq t\leq T}$ the prediction step of a new position. The approximation process $(\bar{X}_t)_{0\leq t\leq T}$ is simply obtained from $(\bar{Y}_t)$ by taking the absolute value of the prediction :

$$\begin{cases} \bar{Y}_{t_{i+1}} = \bar{X}_{t_i} + (t_{i+1} - t_i)\bar{U}_{t_i} \\ \bar{X}_{t_{i+1}} = |\bar{Y}_{t_{i+1}}|. \end{cases} \tag{1.4}$$

A collision of the discretized particle with the wall boundary takes place during the time interval $(t_i, t_{i+1}]$, if

$t_i < t_i - \dfrac{\bar{X}_{t_i}}{\bar{U}_{t_i}} \leq t_{i+1}$. We introduce the sequence of times $(\theta_i, i = 1, \ldots, n)$ defined as

$$\theta_i = \begin{cases} t_i - \dfrac{\bar{X}_{t_i}}{\bar{U}_{t_i}}, & \text{if } t_i < t_i - \dfrac{\bar{X}_{t_i}}{\bar{U}_{t_i}} \leq t_{i+1}, \\ t_i, & \text{otherwise.} \end{cases} \tag{1.5}$$

We call the (θ_i) the collision times (expect when $\theta_i = t_i$), and we observe that when $\theta_i > t_i$,

$$\bar{Y}_{\theta_i} = \bar{X}_{\theta_i} = 0.$$

• **Discretization of the velocity process.**

$$\begin{cases} \text{if } \theta_i \in (t_i, t_{i+1}], \text{ a collision takes place during the interval:} \\ \quad \text{for } t_i \leq t < \theta_i \\ \quad\quad \bar{U}_t = \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(t - t_i) + \sigma(W_t - W_{t_i}) \\ \quad \text{at } \theta_i, \text{ velocity reflection :} \\ \quad\quad \bar{U}_{\theta_i} = -\bar{U}_{\theta_i^-} \\ \quad \text{for } \theta_i < t \leq t_{i+1} : \\ \quad\quad \bar{U}_t = \bar{U}_{\theta_i} + b(\bar{X}_{\theta_i}, \bar{U}_{\theta_i})(t - \theta_i) + \sigma(W_t - W_{\theta_i}) \\ \text{else, no collision :} \\ \quad \text{for } t_i \leq t \leq t_{i+1} \\ \quad\quad \bar{U}_t = \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(t - t_i) + \sigma(W_t - W_{t_i}). \end{cases} \tag{1.6}$$

When $d > 1$, the scheme writes exactly the same, except that one have to adapt the computation of the collision time and the velocity reflection as

$$\theta_i = t_i + \frac{\bar{X}^{(d)}_{t_i}}{(\bar{U}_{t_i} \cdot n_{\mathcal{D}})}$$

and

$$(\bar{U}_{\theta_i} \cdot n_{\mathcal{D}}) = -(\bar{U}_{\theta_i^-} \cdot n_{\mathcal{D}})).$$

Similar schemes to the one presented above have been applied for confined and McKean non linear Lagrangian models involved in the modelling of turbulent atmospheric flow (see [1] and [3]). In particular, particles collisions with the boundary simulation domain are used to impose Dirichlet boundary condition for the velocity. The scheme is also implemented in the WindPos[1] software for wind simulation and wind farms based on fluid particle simulation.

In what follows, we prove the first rate of convergence result for the weak error produce by such scheme.

1.1 Main result

Let us first introduce hereafter our hypotheses. From now on, we implicitly assume that σ is strictly positive. A first set of hypotheses ($H_{Langevin}$) is needed to insure the existence of a solution to the system (1.1). A second set (H_{PDE}) insures the existence and the regularity of a solution to the backward Kolmogorov PDE associated to the SDE (1.1). A third set ($H_{Weak\ Error}$) is added to insure the weak convergence rate of order one.

Hypotheses 1.1

($H_{Langevin}$)-(i) The initial condition (X_0, U_0) is assumed to be distributed according to a given initial law μ_0 having its support in $\mathcal{D} \times \mathbb{R}^d$ and such that $\int_{\mathcal{D}\times\mathbb{R}^d} \left(|x|^2 + |u|^2\right) \mu_0(dx, du) < +\infty$.

($H_{Langevin}$)-(ii) The drift $b \colon \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is uniformly bounded and Lipschitz-continuous with constant $\|b\|_{Lip}$.

(H_{PDE})-(i) The drift b is a $\mathcal{C}^{1,1}_b(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ function, and the first derivatives $\nabla_x b$ and $\nabla_u b$ are also Lipschitz on $\mathbb{R}^d \times \mathbb{R}^d$.

[1] see https://windpos.inria.fr

(H_{PDE})-(ii) When $x \in \partial\mathcal{D}$, the d^{th} coordinate of $u \mapsto b(x,u)$ is an odd function in terms of the d^{th} coordinate of the variable u. The first $(d-1)$ coordinates of $b(x,u)$ (denoted $b'(x,u)$) are even functions with respect to the same d^{th} coordinate of the variable u. In particular, for any $x = (x',0) \in \partial\mathcal{D}$ and $u \in \mathbb{R}^d$,

$$b(x,u) = (b', b^{(d)})((x',0),(u',u^{(d)})) = (b', -b^{(d)})((x',0),(u',-u^{(d)})),$$

where for any vector $v \in \mathbb{R}^d$, v' denotes the $d-1$ firsts components and $v^{(d)}$ denotes the d^{th} one.

($H_{\text{Weak Error}}$) μ_0 admits a Lebesgue density function that is still denoted μ_0 in $L^\infty(\mathcal{D}\times\mathbb{R}^d)$ and there exists $\varepsilon_0 > 0$ such that

$$\frac{\inf\{x; (x,u) \in \mathrm{Supp}(\mu_0)\}}{\inf\{u; (x,u) \in \mathrm{Supp}(\mu_0) \text{ and } u < 0\}} < -\varepsilon_0.$$

Remark 1.2. *The results presented below remain valid if we assume that the drift b is also time dependent with $b \in \mathcal{C}^1((0,T); \mathcal{C}_b^{1,1}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d))$ and $\nabla_x b$, $\nabla_u b$ are Lipschitz.*

Remark 1.3. *The condition (H_{PDE})-(ii) restricts strongly the set of drifts b for which we can claim a first order convergence rate for the weak error. However a typical example of drift b, coming from the application of colloidal particles carrying by a flow, respects this condition. A particle in a flow undergo a drag force that is modeled in the velocity equation as*

$$b(t,x,u) = -k(t,x)(u - \mathcal{V}(t,x)),$$

where $\mathcal{V}(t,x)$ is the velocity of the fluid seen by the particle at position x and at the time t. In a laminar or turbulent flow, a no-permeability condition at the wall is imposed, that implies that for all $x \in \partial\mathcal{D}$,

$$(\mathcal{V}(t,x) \cdot n_{\mathcal{D}}(x)) = 0.$$

In our case of hyperplane $\mathcal{D}$, this means that for $(x,u) \in \partial\mathcal{D} \times \mathbb{R}$, $\mathcal{V}^{(d)}(t,x) = 0$ and

$$b^{(d)}(x,u) = b^{(d)}(x,u^{(d)}) = -k(t,x)\, u^{(d)}.$$

For such important example, for $x \in \partial\mathcal{D}$, $b^{(d)}(x,\cdot)$ is odd in $u^{(d)}$ and the b' components do not depend on $u^{(d)}$ and satisfy (H_{PDE})-(ii).

Remark 1.4. *Later in the proofs, we will introduce again the transformed drift $\widetilde{b}$ used in (1.3) to construct a solution to (1.1) and defined as*

$$\tilde{b} \colon (y,v) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \Big(b', \mathrm{sign}(y^{(d)}) b^{(d)}\Big)\big((y', |y^{(d)}|), (v', \mathrm{sign}(y^{(d)}) v^{(d)})\big).$$

where the function sign *is defined in (1.11). Hypotheses ($H_{Langevin}$)-(ii) and (H_{PDE})-(ii) ensure the continuity of $\widetilde{b}$. Indeed, for $(y,v) \in (\mathbb{R}^d \setminus \partial\mathcal{D}) \times \mathbb{R}^d$, by the hypothesis ($H_{PDE}$)-(i), we have that $\widetilde{b}$ is continuous at (y,v).*

Let $(y,v) \in \partial\mathcal{D} \times \mathbb{R}^d$, then by the evenness condition in (H_{PDE})-(ii), we have that

$$\widetilde{b}'(y,v) = b'((y',0),(v',-v^{(d)})) = b'((y',0),(v',v^{(d)})) = \lim_{h \searrow 0} b((y',h),v) = \lim_{h\searrow 0} \widetilde{b}'((y',h),v).$$

and

$$\widetilde{b}^{(d)}(y,v) = -b^{(d)}((y',0),(v',-v^{(d)})) = b^{(d)}((y',0),(v',v^{(d)})) = \lim_{h\to 0} \widetilde{b}^{(d)}((y',h),v).$$

By ($H_{Langevin}$)-(ii), $\widetilde{b}$ is also piecewise Lipschitz. Together with the continuity property, $\widetilde{b}$ is uniformly Lipschitz with a Lipschitz constant $\|\tilde{b}\|_{Lip}$ equal to $2\|b\|_{Lip}$.

Indeed, for $i = 1,\dots,d-1$,

$$\begin{aligned}
&|\tilde{b}^{(i)}(x,u) - \tilde{b}^{(i)}(y,v)| \\
&= |(b')^{(i)}\big((x',|x^{(d)}|),(u',\mathrm{sign}(x^{(d)})u^{(d)})\big) - (b')^{(i)}\big((y',|y^{(d)}|),(v',\mathrm{sign}(y^{(d)})v^{(d)})\big)| \\
&\le \mathbb{1}_{\{\mathrm{sign}(x^{(d)}y^{(d)})=1\}} \{\|b\|_{\mathrm{Lip}} (\|x-y\| + \|u-v\|)\} \\
&\quad + \mathbb{1}_{\{\mathrm{sign}(x^{(d)}y^{(d)})=-1\}} \left|(b')^{(i)}\big((y',|y^{(d)}|),(v',\mathrm{sign}(y^{(d)})v^{(d)})\big) - (b')^{(i)}\big(0,(v',\mathrm{sign}(y^{(d)})v^{(d)})\big)\right| \\
&\quad + \mathbb{1}_{\{\mathrm{sign}(x^{(d)}y^{(d)})=-1\}} \left|(b')^{(i)}\big(0,(v',\mathrm{sign}(y^{(d)})v^{(d)})\big) - (b')^{(i)}\big(0,(u',\mathrm{sign}(x^{(d)})u^{(d)})\big)\right| \\
&\quad + \mathbb{1}_{\{\mathrm{sign}(x^{(d)}y^{(d)})=-1\}} \left|(b')^{(i)}\big((x',|x^{(d)}|),(u',\mathrm{sign}(x^{(d)})u^{(d)})\big) - (b')^{(i)}\big(0,(u',\mathrm{sign}(x^{(d)})u^{(d)})\big)\right|.
\end{aligned}$$

Using hypothesis (H_{PDE})-(ii), the third term above is bounded by $\|b^{(i)}\|_{\text{Lip}}\|u-v\|$. Moreover, since $\mathbb{1}_{\{\text{sign}(x^{(d)}y^{(d)})=-1\}}(\|x\|+\|y\|) \leq 2\|x-y\|$, we conclude that $|\tilde{b}^{(i)}(x,u) - \tilde{b}^{(i)}(y,v)| \leq 2\|b^{(i)}\|_{\text{Lip}}(\|x-y\| + \|u-v\|)$. Similarly, for the d-component, using several times that for any (x,y,u,v),

$$\mathbb{1}_{\{\text{sign}(x^{(d)}y^{(d)})=-1\}} \Big(\text{sign}(y^{(d)})b^{(d)}\big(0,(v',\text{sign}(y^{(d)})v^{(d)})\big) - \text{sign}(x^{(d)})b^{(d)}\big(0,(v',\text{sign}(x^{(d)})v^{(d)})\big)\Big) = 0,$$

we obtain with the same decomposition that as well that,

$$|\tilde{b}^{(d)}(x,u) - \tilde{b}^{(d)}(y,v)| \leq 2\|b^{(d)}\|_{\text{Lip}}(\|x-y\| + \|u-v\|).$$

Remark 1.5. *The condition ($H_{\text{Weak Error}}$) on the support of μ_0 implies that the first collision time of the scheme (1.4)-(1.6) is almost surely separated from $t = 0$. In the proposed scheme, the first possible collision time before Δt is*

$$-\frac{X_0}{U_0} \geq \varepsilon_0 > 0.$$

Rate of convergence result

We denote by Q_T the set $(0,T) \times \mathcal{D} \times \mathbb{R}^d$. For any measurable function ψ defined on $\overline{\mathcal{D}} \times \mathbb{R}^d$, we consider the function $F : Q_T \longrightarrow \mathbb{R}$ defined as

$$F(t,x,u) = \mathbb{E}\psi(X_T^{t,x,u}, U_T^{t,x,u}) \tag{1.7}$$

where the process $(X_s^{t,x,u}, U_s^{t,x,u})_{s\geq t}$ solves the SDE (1.1) that begins at time t with values (x,u).

Our main result is the following

Theorem 1.6. *Assume ($H_{Langevin}$), (H_{PDE}) and ($H_{\text{Weak Error}}$) and fix $T > 0$. Then, for any test function $\psi \in \mathcal{C}_c^{1,1}(\mathcal{D},\mathbb{R}^d;\mathbb{R})$, there exists a constant C_{F,σ,b,T,μ_0} such that we can prove a first order convergence bound for the weak approximation error:*

$$\left|\mathbb{E}\psi(X_T,U_T) - \mathbb{E}\psi(\bar{X}_T,\bar{U}_T)\right| \leq C_{F,\sigma,b,T,\mu_0}\,\Delta t \tag{1.8}$$

where C_{F,σ,b,T,μ_0} depends only on the solution F to the PDE (1.9) and their derivatives, on the drift b and their derivatives, on the diffusion constant σ, on the terminal time T and on the norm $\|\mu_0\|_{L^\infty}$ of the initial density distribution of (X_0,U_0).

A key argument in the proof of the theorem resides in the regularity we can show for the function F.

We start, showing first in section 6 that when ψ is in $\mathcal{C}_c(\mathcal{D},\mathbb{R})$, F is a weak solution to the following backward Kolmogorov PDE (see Proposition 6.4) with specular boundary condition:

$$\begin{cases} \partial_t F + (u\cdot\nabla_x F) + (b(x,u)\cdot\nabla_u F) + \dfrac{\sigma^2}{2}\triangle_u F = 0, \text{ on } Q_T, \\ F(T,x,u) = \psi(x,u), \text{ on } \mathcal{D}\times\mathbb{R}^d, \\ F(t,x,u) = F(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \text{ on } \Sigma_T^+. \end{cases} \tag{1.9}$$

with Σ_T^+ defined in (1.10). A priori L^2 bound for first order derivatives of F is shown in Section 4. The proof of this result is based on the probabilistic expression of F in (1.7). Section 5 is dedicated to higher order regularity result using L^2 energy inequality formulation. Furthermore, in section 4 we show that the first derivatives are in $L^\infty(Q_t)$.

Section 2 presents a schematic proof of the weak error rate in the case of a diffusion without any boundaries. We also introduce some results needed for the proof of the main theorem. The proof of Theorem 1.6 is given in section 3 and is based on regularity obtained on F.

In order to simplify notations, the analysis for Section 3 is given assuming $d = 1$. In the other sections, the dimension d is arbitrary, unless it is explicitly mentioned.

1.2 Notation

The space $\mathcal{C}_b^{l,m}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ is the set of continuous and bounded functions on $\mathbb{R}^d \times \mathbb{R}^d$, with continuous and bounded derivatives with respect to the variables in $\mathbb{R}^d \times \mathbb{R}^d$, up to the order l and m respectively.

The space $\mathcal{C}_c^{l,m}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ has the same definition but for functions with compact supports.

The space $\mathcal{C}_c^{l}(\mathbb{R}^d)$ is the set of continuous functions on $\mathbb{R}^d$ with compact supports, with continuous and bounded derivatives up to the order l.

For all $t \in (0,T]$, we introduce the time-phase space

$$Q_t := (0,t) \times \mathcal{D} \times \mathbb{R}^d,$$

the outward normal to $\mathcal{D}$ noted by $n_{\mathcal{D}}$ and the boundary sets:

$$\begin{aligned}
&\Sigma^+ := \big\{(x,u) \in \partial\mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}(x)) > 0\big\}, && \Sigma_t^+ := (0,t) \times \Sigma^+,\\
&\Sigma^- := \big\{(x,u) \in \partial\mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}(x)) < 0\big\}, && \Sigma_t^- := (0,t) \times \Sigma^-,\\
&\Sigma^0 := \big\{(x,u) \in \partial\mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}(x)) = 0\big\}, && \Sigma_t^0 := (0,t) \times \Sigma^0,
\end{aligned} \tag{1.10}$$

and further $\Sigma_T := \Sigma_T^+ \cup \Sigma_T^0 \cup \Sigma_T^- = (0,T) \times \partial\mathcal{D} \times \mathbb{R}^d$. Denoting by $d\sigma_{\partial\mathcal{D}}$ the surface measure on $\partial\mathcal{D}$, we introduce the product measure on Σ_T:

$$d\lambda_{\Sigma_T} := dt \otimes d\sigma_{\partial\mathcal{D}}(x) \otimes du.$$

We introduce the Sobolev space

$$\mathcal{H}(Q_t) = L^2((0,t) \times \mathcal{D}; H^1(\mathbb{R}^d))$$

equipped with the norm $\| \, \|_{\mathcal{H}(Q_t)}$ defined by

$$\|\phi\|^2_{\mathcal{H}(Q_t)} = \|\phi\|^2_{L^2(Q_t)} + \|\nabla_u \phi\|^2_{L^2(Q_t)}.$$

We denote by $\mathcal{H}'(Q_t)$, the dual space of $\mathcal{H}(Q_t)$, and by $(\ ,\ )_{\mathcal{H}'(Q_t),\mathcal{H}(Q_t)}$, the inner product between $\mathcal{H}'(Q_t)$ and $\mathcal{H}(Q_t)$.

We further introduce the space

$$L^2(\Sigma_T^\pm) = \Big\{\psi : \Sigma_T^\pm \to \mathbb{R} \text{ s.t } \int_{\Sigma_T^\pm} |(u \cdot n_{\mathcal{D}}(x))| \, |\psi(t,x,u)|^2 \, d\lambda_{\Sigma_T}(t,x,u) < +\infty\Big\},$$

equipped with the norm

$$\|\psi\|_{L^2(\Sigma_T^\pm)} = \sqrt{\int_{\Sigma_T^\pm} |(u \cdot n_{\mathcal{D}}(x))| \ |\psi(t,x,u)|^2 \, d\lambda_{\Sigma_T}(t,x,u)}.$$

The space $L^2(\Sigma_T)$ is defined, through the respective restriction on $\Sigma_T^\pm$ denoted $|_{\Sigma_T^\pm}$ as

$$L^2(\Sigma_T) = \Big\{\psi : \Sigma_T \to \mathbb{R} \text{ s.t } \psi|_{\Sigma_T^\pm} \in L^2(\Sigma_T^\pm)\Big\},$$

and equipped with the norm

$$\|\psi\|_{L^2(\Sigma_T)} = \ \ \|\psi|_{\Sigma_T^+}\|_{L^2(\Sigma_T^+)} + \|\psi|_{\Sigma_T^-}\|_{L^2(\Sigma_T^-)}.$$

The following convention for the function sign: $x \in \mathbb{R} \mapsto \mathbb{R}$ is considered:

$$\operatorname{sign}(x) = \begin{cases} -1, & \text{for } x \le 0 \\ \ \ 1, & \text{for } x > 0 \end{cases} \tag{1.11}$$

For multidimensional functions, we use the following definition of L^2 space:

$$L^2(Q_T; \mathbb{R}^d) = \Big\{\psi : Q_T \to \mathbb{R}^d \text{ s.t } \int_{Q_T} \|\psi\|^2 < +\infty\Big\},$$

where $\|\|$ is the Euclidean norm on $\mathbb{R}^d$.

$$L^2(Q_T;\mathbb{R}^{d\times d}) = \big\{\psi : Q_T \to \mathbb{R}^{d\times d} \text{ s.t } \int_{Q_T} \|\psi\|_F^2 < +\infty\big\},$$

where $\|\|_F$ is the Frobenius norm i.e. for any $d\times d$ matrix A, $\|A\|_F = \sqrt{\mathrm{Tr}(AA^T)} = \sqrt{\sum_{i=1}^d \sum_{j=1}^d |a_{ij}|^2}$.

We denote by $\mathrm{Jac}_x(\psi) = \left(\dfrac{\partial \psi_i}{\partial x_j}\right)_{1\le i,j\le d}$ the Jacobian matrix of $\psi : \mathbb{R}^d \to \mathbb{R}^d$ w.r.t x and $\mathrm{Hess}_{x,u}(\varphi) = \left(\dfrac{\partial^2\varphi}{\partial x_i \partial u_j}\right)_{1\le i,j\le d}$ is the Hessian matrix w.r.t (x,u) of $\varphi : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$.

For any functions $G\colon Q_T \mapsto \mathbb{R}$, $\gamma_1 : \mathbb{R}\mapsto\mathbb{R}$, $\gamma_2 : \mathbb{R}^d \mapsto \mathbb{R}$ and $\gamma_3 : \mathbb{R}^d \mapsto \mathbb{R}$, we define the joint convolution $G*(\gamma_1\gamma_2\gamma_3)$ at any $(t,x,u) \in \overline{Q_T}$ as :

$$G*(\gamma_1\gamma_2\gamma_3)(t,x,u) := \int_{Q_T} G(\tau,y,v)\gamma_1(t-\tau)\gamma_2(x-y)\gamma_3(u-v)\, d\tau dy dv\,.$$

In case of multi-dimensional functions, the convolution applies on each components.

We will denote by $\|f\|_{\mathrm{Lip}}$ the Lipschitz constant of a function f from $\mathbb{R}^d$ to $\mathbb{R}^d$, defined as the smaller constant C such that

$$\|f(u) - f(u')\| \le C\|u-u'\|.$$

For a mapping $\mathbb{R}^d\times\mathbb{R}^d \ni (x,u) \to f(x,u) \in \mathbb{R}^d$, we denote by $\|f\|_{\infty_x,\mathrm{Lip}_u}$, Lipschitz constant of f with respect to u, uniformly on x, defined as

$$\|f\|_{\infty_x,\mathrm{Lip}_u} = \sup_{x\in\mathbb{R}^d} \|f(x,\cdot)\|_{\mathrm{Lip}}.$$

2 Preliminaries

We present a schematic of the usual method to obtain the weak error convergence rate. Let's consider a process $(Z_t)_{0\le t\le T}$, defined on $\mathbb{R}$, that is simple and unconfined SDE:

$$dZ_t = b(Z_t)\,dt + \sigma\, dW_t$$

where b is a sufficiently smooth bounded function. It is well known (see e.g [14]) that, for any ψ in $C_b^2(\mathbb{R})$, there exists a classical solution $g \in \mathcal{C}_b^{1,2}((0,T)\times\mathbb{R})$ to the backward PDE:

$$\begin{cases} \dfrac{\partial g}{\partial t} + b(z)\dfrac{\partial g}{\partial z} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 g}{\partial z^2} = 0 \\ g(T,z) = \psi(z) \quad \forall z\in\mathbb{R}, \end{cases}$$

such that $g(t,z) = \mathbb{E}\psi(Z_T^{t,z})$, where $(Z_\theta^{t,z}, \theta > t)$ is the flow solution starting from the point $Z_t^{t,z} = z$. We denote by $\mathcal{L}$ the infinitesimal generator of the process $(Z_t)_{t\ge 0}$ defined for any $h\in\mathcal{C}^2$ by:

$$\mathcal{L}h(z) = b(z)\frac{\partial h}{\partial z}(z) + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial z^2}(z).$$

We introduce a regular time grid $0 = t_0 < t_1 < \ldots < t_n = T$, and the corresponding times-freezing function $\eta\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ defined as $\eta(t) = t_i$ when $t\in[t_i,t_{i+1})$. We consider the continuous version $(\bar{Z}_t)_{t\ge 0}$ of the Euler scheme applied to Z as:

$$\bar{Z}_t = Z_0 + \int_0^t b(\bar{Z}_{\eta(s)})\,ds + \sigma\, W_t.$$

Now for any $\bar{z}\in\mathbb{R}$, we consider also $\mathcal{L}^{\bar{z}}$ the differential operator defined also on $\mathcal{C}^2$ functions by:

$$\mathcal{L}^{\bar{z}}h(z) = b(\bar{z})\frac{\partial h}{\partial z}(z) + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial z^2}(z).$$

The weak error produced by the Euler scheme for the test function ψ can be obtained by applying the Itô's formula two successive times. From a first application, we get for a fixed $z \in \mathbb{R}$, since $g(0,z) = \mathbb{E}\psi(Z_T^{0,z})$,

$$\begin{aligned}\mathbb{E}\psi(\bar{Z}_T^{0,z}) - \mathbb{E}\psi(Z_T^{0,z}) &= \mathbb{E}\left[g(T,\bar{Z}_T^{0,z}) - g(0,z)\right] \\ &= \mathbb{E}\int_0^T \left(\partial_t g(t,\bar{Z}_t) + \mathcal{L}^{\bar{Z}_{\eta(t)}} g(t,\bar{Z}_t)\right) dt =\end{aligned}$$

Since $\partial_t g + \mathcal{L}g = 0$, the previous equality becomes

$$\mathbb{E}\psi(\bar{Z}_T^{0,z}) - \mathbb{E}\psi(Z_T^{0,z}) = \mathbb{E}\int_0^T \left(\mathcal{L}^{\bar{Z}_{\eta(t)}} g(t,\bar{Z}_t) - \mathcal{L}g(t,\bar{Z}_t)\right) dt = \mathbb{E}\int_0^T \frac{\partial g}{\partial z}(t,\bar{Z}_t)\left(b(\bar{Z}_{\eta(t)}) - b(\bar{Z}_t)\right) dt$$

Now observing that for every time step t_i, we have that $\mathcal{L}^{\bar{Z}_{t_i}} g(t_i,\bar{Z}_{t_i}) = \mathcal{L}g(t_i,\bar{Z}_{t_i})$, by applying the Itô's formula once more on the interval $[\eta(t),t)$, we get

$$\begin{aligned}\mathbb{E}\psi(\bar{Z}_T^{0,z}) - \mathbb{E}\psi(Z_T^{0,z}) &= \mathbb{E}\int_0^T dt \int_{\eta(t)}^t ds \left(\mathcal{L}^{\bar{Z}_{\eta(t)}}\left(\frac{\partial g}{\partial z}(s,\bar{Z}_s)\left(b(\bar{Z}_{\eta(t)}) - b(\bar{Z}_s)\right)\right)\right) \\ &\quad + \mathbb{E}\int_0^T dt \int_{\eta(t)}^t ds \left(\frac{\partial}{\partial s}\frac{\partial g}{\partial z}(s,\bar{Z}_s)\left(b(\bar{Z}_{\eta(t)}) - b(\bar{Z}_s)\right)\right).\end{aligned}$$

Since g has bounded derivatives, the stochastic integrals from the applications of the Itô's formula are martingales.

The Δt factor, for the weak error convergence, is then extracted from the inner integral, since for any $t \in [0,T]$, $|t - \eta(t)| \leq \Delta t$. If b is in $C_b^2(\mathbb{R})$ then there exists a constant K_T which depends on T such that for all $n = 0,1,2$, $|\partial_z^n g(t,z)| < K_T \|\psi\|_{W^{3,\infty}}$. This can be proven directly from $g(t,z) = \mathbb{E}\psi(Z_T^{t,z})$. Then, the previous equality can be bounded by

$$\left|\mathbb{E}\psi(\bar{Z}_T^{0,z}) - \mathbb{E}\psi(Z_T^{0,z})\right| \leq C_{\partial^\alpha\varphi,\partial^\alpha b,\sigma,T}\Delta t$$

where $C_{\partial^\alpha\varphi,\partial^\beta b,\sigma,T}$ depends only on bounds for the derivatives of ψ up to the order 3, derivatives of b up to the order 2.

The proof of Theorem 1.6 is build on the same arguments, with certain particular differences that need to be adapted suitably:

- In Section 5, we prove that the solution to the Kolmogorov PDE (1.9) has some regularity in the $L^2(Q_T)$ space (see Theorem 2.1), instead of in $L^\infty(Q_T)$ space as in the previous sketch. Therefore the distribution of the initial values will be used to make appear L^2 norms in the previous arguments.
- Also, since we are interested in a confined SDE and backward PDE with specular condition, we will have to take into consideration boundary effects and adapt the form of the continuous version of the time discretization scheme.
- In order to apply Ito's formula as previously used, a time-continuous version of the schemes (1.4) and (1.6) need to be introduced. For this we consider first the function $\eta\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ defined as previously as

  $$\eta(t) = t_i, \quad \forall t \in [t_i, t_{i+1}).$$

  Second, recalling the definition of the collision times in (1.5), we introduce $\nu\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ defined as:

  $$\nu(t) = \begin{cases} t_i & \text{for } t_i \leq t < \theta_i \\ \theta_i & \text{for } \theta_i \leq t < t_{i+1}. \end{cases} \tag{2.1}$$

  We recall that θ_i is meant to signal if a collision is to take place on the interval $[t_i, t_{i+1})$. If there is a collision on this interval, then ν is t_i before the collision and θ_i after. If no collision takes place then ν is t_i.

  With the help of $t \mapsto \eta(t)$ and $t \mapsto \nu(t)$, we write the continuous version of the discrete process as:

  $$\begin{cases} \bar{Y}_t = \bar{X}_{\eta(t)} + (t - \eta(t))\bar{U}_{\eta(t)} \\ \bar{X}_t = X_0 + \displaystyle\int_0^t \bar{U}_{\eta(s)} \operatorname{sign}(\bar{Y}_s)\, ds \\ \bar{U}_t = U_0 + \displaystyle\int_0^t b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)})\, ds + \sigma W_t - 2\sum_{0<s\leq t} \bar{U}_{s^-}\mathbb{1}_{\bar{X}_s=0}. \end{cases} \tag{2.2}$$

2.1 The backward Kolmogorov PDE

We give some regularity results on the solution of the PDE (1.9).

Theorem 2.1. *Assume (H_{PDE}). When ψ belongs in $\mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d;\mathbb{R})$, F defined in (1.7) belongs in $\mathcal{H}(Q_T)$, and is solution in the sense of distribution to the backward PDE:*

$$\begin{cases} \partial_t F + (u\cdot\nabla_x F) + (b(x,u)\cdot\nabla_u F) + \dfrac{\sigma^2}{2}\triangle_u F = 0, \text{ on } Q_T, \\ F(T,x,u) = \psi(x,u), \text{ on } \mathcal{D}\times\mathbb{R}^d, \\ F(t,x,u) = F(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \text{ on } \Sigma_T^+. \end{cases} \tag{1.9 bis}$$

When $\psi\in\mathcal{C}_c^{1,1}(\mathcal{D}\times\mathbb{R}^d;\mathbb{R})$, then F is in $\mathcal{C}([0,T];L^\infty(\mathcal{D}\times\mathbb{R}^d);\mathbb{R}^d)\cap\mathcal{C}([0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d;\mathbb{R})\cap L^2(Q_T;\mathbb{R}^d)$. The derivatives $\nabla_x F$ and $\nabla_u F$ exist and belong in $\mathcal{C}([0,T];L^\infty(\mathcal{D}\times\mathbb{R}^d);\mathbb{R}^d)\cap\mathcal{C}([0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d;\mathbb{R}^d)\cap L^2(Q_T;\mathbb{R}^d)$. By continuity up to $\partial\mathcal{D}$, a trace on Σ_T exists for those functions in $L^2(\Sigma_T;\mathbb{R}^d)$.

Moreover $Hess_{x,u}(F), Hess_{u,u}(F)\in L^2(Q_T;\mathbb{R}^{2d})$.

The proof of Theorem 2.1 is divided in the three following sections:

- We prove that F has derivatives w.r.t. x and u that can be extended up to the boundary Σ_T and have finite $L^2(\Sigma_T)$ norm, we will make use of the probabilistic form of F in (1.7). In section 4, we show the regularity of the flow of the free Lagrangian process (in the sens of Bouleau Hirsch) and apply this result to prove the existence of the first order derivatives of F (see Lemma 4.6).
- In section 5, we show the L^2 regularity of the Hessians of F using a variational approach on the PDE (1.7) (see Corollary 5.5).
- In section 6, we extend some results of [6] on the semi group of the confined Langevin process with a drift.

2.2 Begining of the proof of main Theorem 1.6

Let us start with the weak error term

$$\mathbb{E}\psi(X_T^{X_0,U_0},U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0},\bar{U}_T^{X_0,U_0})$$

for a given test function ψ.

From the definition of the function F in (1.7), we have

$$\begin{aligned} \mathbb{E}\psi(X_T^{X_0,U_0},U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0},\bar{U}_T^{X_0,U_0}) &= \mathbb{E}F(0,X_0,U_0) - \mathbb{E}F(T,\bar{X}_T^{X_0,U_0},\bar{U}_T^{X_0,U_0}) \\ &= \mathbb{E}\sum_{i=0}^{n-1}\left(F(t_i,\bar{X}_{t_i}^{X_0,U_0},\bar{U}_{t_i}^{X_0,U_0}) - F(t_{i+1},\bar{X}_{t_{i+1}}^{X_0,U_0},\bar{U}_{t_{i+1}}^{X_0,U_0})\right) \\ &= \mathbb{E}\sum_{i=0}^{n-1}\left(F(t_i,\bar{X}_{t_i}^{X_0,U_0},\bar{U}_{t_i}^{X_0,U_0}) - F(t_{i+1}^-,\bar{X}_{t_{i+1}^-}^{X_0,U_0},\bar{U}_{t_{i+1}^-}^{X_0,U_0})\right). \end{aligned}$$

Let us explain the last equality. The function F is continuous with respect to its three variables (t,x,u) (see Lemma 4.5). So if t_{i+1} is not a collision instant, then the scheme $(\bar{X}_t,\bar{U}_t)_{0\le t\le T}$ is continuous as time t_{i+1}, so the passage from the second to the third line in the previous equality is obvious. If at t_{i+1} a collision takes place, then

$$\bar{X}_{t_{i+1}^-}^{X_0,U_0} = \bar{X}_{t_{i+1}}^{X_0,U_0} = 0$$

and

$$\bar{U}_{t_{i+1}}^{X_0,U_0} = -\bar{U}_{t_{i+1}^-}^{X_0,U_0},$$

and since F satisfies the boundary specular condition, then we obtain once more the equality.

Now the collision times are introduced via the function $t \mapsto \nu(t)$ in (2.1) as follows:

$$\begin{aligned}
&\mathbb{E}\psi(X_T^{X_0,U_0}, U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0}, \bar{U}_T^{X_0,U_0}) \\
&= \mathbb{E}\sum_{i=0}^{n-1}\left(F(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0}) - F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)^-}^{X_0,U_0})\right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1}\left(F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)^-}^{X_0,U_0}) - F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0})\right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1}\left(F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0}) - F(t_{i+1}^-, \bar{X}_{t_{i+1}^-}^{X_0,U_0}, \bar{U}_{t_{i+1}^-}^{X_0,U_0})\right).
\end{aligned}$$

From the definition (2.1), $\nu(t_{i+1}^-) = t_i$ if there is no collision inside the period (t_i, t_{i+1}), otherwise $\nu(t_{i+1}^-) = \theta_i \neq t_i$. If no collision takes place, then by the continuity of F the first two sums of the r.h.s. are zero. If a collision does take place, then by the specular condition on F, the second term of the r.h.s. is zero. So the previous equality becomes:

$$\begin{aligned}
&\mathbb{E}\psi(X_T^{X_0,U_0}, U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0}, \bar{U}_T^{X_0,U_0}) \\
&= \mathbb{E}\sum_{i=0}^{n-1}\left(F(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0}) - F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)^-}^{X_0,U_0})\right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1}\left(F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0}) - F(t_{i+1}^-, \bar{X}_{t_{i+1}^-}^{X_0,U_0}, \bar{U}_{t_{i+1}^-}^{X_0,U_0})\right).
\end{aligned} \tag{2.3}$$

The first sum in the r.h.s can be seen as the contribution to the error of the discretized process before the jump on the time-step $[t_i, t_{i+1}]$, while the second sum is the contribution to the error of the process after the collision.

We continue the proof of the main theorem in Section 3, with the help of Theorem 2.1.

Before that, we end this section with the estimation of a bound for the L^∞ norm of the density of the confined time discretized process. In [5], it is shown that the confined Lagrangian process (1.1) admits an explicit density. Following the same arguments, we exhibit a transition density for the discretized confined Brownian primitive (i.e. $b \equiv 0$):

Lemma 2.2. *Under ($H_{\text{Weak Error}}$), the process solution to the system (2.2) with drift $b \equiv 0$ has a bounded density $p^c(t, \zeta, \eta)$, bounded by $2\|\mu_0\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d)}$.*

Proof. Starting from a given (x, u) in $\mathcal{D} \times \mathbb{R}^d$, the process (2.2) without drift can be written as:

$$\begin{cases}
\bar{x}_t = x + \displaystyle\int_0^t \bar{u}_{\eta(s)} \operatorname{sign}\left(\bar{x}_{\eta(s)} + (s - \eta(s))\bar{u}_{\eta(s)}\right) ds \\
\bar{u}_t = u + \sigma W_t^0 - 2\displaystyle\sum_{0<s\leq t} \bar{u}_{s^-} \mathbb{1}_{\bar{x}_s = 0}
\end{cases} \tag{2.4}$$

where $(W_t^0)_{t\geq 0}$ is a standard Brownian motion. Following the arguments in [5], we introduce the continuous time-discretized free Langevin process with $b = 0$:

$$\begin{cases}
\bar{Z}_t = x + \displaystyle\int_0^t \bar{V}_{\eta(s)}\, ds \\
\bar{V}_t = u + \sigma W_t.
\end{cases} \tag{2.5}$$

The position process $\bar{Z}_t$ can be rewritten as

$$\bar{Z}_t = x + ut + \sigma \sum_{i\geq 0} W_{t_i\wedge t}(t_{i+1}\wedge t - t_i \wedge t).$$

Since $(W_t)_{t\geq 0}$ is a Gaussian process, then $(\bar{Z}_t, \bar{V}_t)_{t\geq 0}$ is also a Gaussian process due to the fact that it can be written as a linear combination of random variables sampled from a Gaussian process at different instants. In particular, there is a Gaussian transition density for the time-discretized Langevin process with no drift, denoted as $\bar{p}^L$ (see Section B for the explicit expression for $\bar{p}^L$.)

Define $S_t = \text{sign}(Z_t)_+$ to be the càdlàg modification of the process $(\text{sign}(Z_t))_{0\le t\le T}$ and set

$$(\bar{X}^c_t, \bar{U}^c_t) = (|\bar{Z}_t|, S_t \bar{V}_t). \tag{2.6}$$

Then, by the Itô's formula, we get

$$\bar{U}^c_t = u + \int_0^t S_{s^-}\, d\bar{V}_s + \sum_{0<s\le t} \bar{V}_s \Delta S_s = u + \sigma \int_0^t S_{s^-}\, dW_s + \sum_{0<s\le t} \bar{V}_s \Delta S_s.$$

Since $\langle \int_0^{\cdot} S_{s^-}\, dW_s\rangle_t = t$, by Lévy's representation theorem, the process $(W^c_t = \int_0^t S_{s^-}\, dW_s, t\ge 0)$ is a Brownian motion. Also, by continuity of the process $(\bar{V}_t)_{0\le t\le T}$, for any $t\in[0,T]$, $\bar{U}^c_{t^-} = \bar{V}_{t^-} S_{t^-} = \bar{V}_t S_{t^-}$. Consider a time interval $[t_i, t_{i+1}]$ such that $t_i < \theta_i < t_{i+1}$, then if $S_{\eta(t)} > 0$, then $S_{\theta_i^-} > 0$ implying that $\Delta S_{\theta_i} = -2 = -2S_{\theta_i^-}$ and if $S_{\eta(t)} < 0$, then $S_{\theta_i^-} < 0$ resulting in $\Delta S_{\theta_i} = 2 = -2S_{\theta_i^-}$. These considerations give that $\bar{V}_{\theta_i}\Delta S_{\theta_i} = -2U^c_{\theta_i^-}$, and finally, we have that

$$\bar{U}^c_t = u + \sigma W^c_t - 2\sum_{0<s\le t} \bar{U}^c_{s^-} \mathbb{1}_{\{\bar{X}^c_s = 0\}}.$$

Considering that $(Z^c_t)_{0\le t\le T}$ change it sign a finite number of time, it admits a regularity C^1 by parts. We obtain that

$$\bar{X}^c_t = |\bar{Z}_t| = x + \int_0^t \text{sign}(\bar{Z}_s)\bar{V}_{\eta(s)}\, ds.$$

From (2.5), we notice that

$$\bar{Z}_t = \bar{Z}_{\eta(t)} + (t-\eta(t))\bar{V}_{\eta(t)} = \text{sign}(\bar{Z}_{\eta(t)})\left(|\bar{Z}_{\eta(t)}| + (t-\eta(t))\,\text{sign}(\bar{Z}_{\eta(t)})\bar{V}_{\eta(t)}\right),$$

since $\text{sign}(ab) = \text{sign}(a)\,\text{sign}(b)$. So,

$$\bar{Z}_t = S_{\eta(t)}\left(\bar{X}^c_{\eta(t)} + (t-\eta(t))\bar{U}^c_{\eta(t)}\right)$$

and

$$\begin{aligned}
\bar{X}^c_t &= x + \int_0^t \text{sign}\left(S_{\eta(s)}\left(\bar{X}^c_{\eta(s)} + (s-\eta(s))\bar{U}^c_{\eta(s)}\right)\right)\bar{V}_{\eta(s)}\, ds \\
&= x + \int_0^t \text{sign}\left(\bar{X}^c_{\eta(s)} + (s-\eta(s))\bar{U}^c_{\eta(s)}\right) S_{\eta(s)}\bar{V}_{\eta(s)}\, ds \\
&= x + \int_0^t \text{sign}\left(\bar{X}^c_{\eta(s)} + (s-\eta(s))\bar{U}^c_{\eta(s)}\right) \bar{U}^c_{\eta(s)}\, ds
\end{aligned}$$

obtaining finally:

$$\begin{cases}
\bar{X}^c_t = x + \int_0^t \text{sign}\left(\bar{X}^c_{\eta(s)} + (s-\eta(s))\bar{U}^c_{\eta(s)}\right) \bar{U}^c_{\eta(s)}\, ds \\
\bar{U}^c_t = u + \sigma W^c_t - 2\sum_{0<s\le t} \bar{U}^c_{s^-}\mathbb{1}_{\{\bar{X}^c_s = 0\}}.
\end{cases}$$

This shows that $(\bar{X}^c_t, \bar{U}^c_t)_{0\le t\le T}$ defined as (2.6) is equal in law to the solution of (2.4) $(\bar{x}_t, \bar{u}_t)_{0\le t\le T}$. This also implies that $(|\bar{u}_t|)_{t\ge 0}$ is equal in distribution to $(|u + W_t|)_{t\ge 0}$. Furthermore, for any measurable and bounded function $h\colon \mathbb{R}^+\times\mathbb{R} \longrightarrow \mathbb{R}$:

$$\mathbb{E}h(\bar{x}_t, \bar{u}_t) = \mathbb{E}\left(h(\bar{Z}_t, \bar{V}_t)\mathbb{1}_{\{\bar{Z}_t>0\}}\right) + \mathbb{E}\left(h(-\bar{Z}_t, -\bar{V}_t)\mathbb{1}_{\{\bar{Z}_t<0\}}\right).$$

as $\{\bar{Z}_t = 0\}$ is negligible. The transition density of the discretized reflected process $\bar{p}^c\colon (\mathbb{R}^+\times(\mathbb{R}^+\times\mathbb{R}))\times(\mathbb{R}^+\times(\mathbb{R}^+\times\mathbb{R})) \to \mathbb{R}$ is then equal to

$$\bar{p}^c(0,x,u;t;\xi,\zeta) = \bar{p}^L(0,x,u;t;\xi,\zeta) + \bar{p}^L(0,x,u;t;-\xi,-\zeta)$$

where $\bar{p}^L$ is the transition density of the time-discretized free process (2.5) computed in Lemma B.1 of the appendix section B.

Now we consider the hypothesis ($H_{Weak\ Error}$), and μ_0 the density of the initial random variable (X_0, U_0). The density of $(\bar{x}_t^{X_0,U_0}, \bar{u}_t^{X_0,U_0})_{0\leq t\leq T}$ writes

$$
\begin{aligned}
p^c(t;\xi,\zeta) &= \int_{\mathbb{R}\times\mathbb{R}^+} \bar{p}^c(0;x,u;t;\xi,\zeta)\mu_0(x,u)\,dxdu \\
&= \int_{\mathbb{R}\times\mathbb{R}^+} \left(\bar{p}^L(0;x,u;t;\xi,\zeta) + \bar{p}^L(0;x,u;t;-\xi,-\zeta)\right)\mu_0(x,u)\,dxdu \\
&= \int_{\mathbb{R}\times\mathbb{R}^+} \Big(p_{\mathcal{N}(0,\Sigma_{t,\Delta t,\eta(t)})}(\xi-(x+tu),\zeta-u) + p_{\mathcal{N}(0,\Sigma_{t,\Delta t,\eta(t)})}(-\xi-(x+tu),-\zeta-u)\Big)\mu_0(x,u)\,dxdu,
\end{aligned}
$$

where $p_{\mathcal{N}(0,\Sigma_{t,\Delta t,\eta(t)})}$ denotes the centered Gaussian density with covariance $\Sigma_{t,\Delta t,\eta(t)}$ computed in Lemma B.1. Then

$$
\begin{aligned}
&p^c(t;\xi,\zeta) \\
&\leq \|\mu_0\|_{L^\infty(\mathcal{D}\times\mathbb{R})}\int_{\mathbb{R}\times\mathbb{R}}\Big(p_{\mathcal{N}(0,\Sigma_{t,\Delta t,\eta(t)})}(\xi-(x+tu),\zeta-u) + p_{\mathcal{N}(0,\Sigma_{t,\Delta t,\eta(t)})}(-\xi-(x+tu),-\zeta-u)\Big)\,dxdu \\
&\leq 2\,\|\mu_0\|_{L^\infty(\mathcal{D}\times\mathbb{R})}.
\end{aligned}
$$

□

3 Weak error estimation

In this section we prove the main theorem 1.6. In order to simplify the presentation, we give the proof for the dimension $d = 1$ and in order to better understand the various definitions for the errors that have been introduced we refer to the diagram 2 in the Appendix section A.

The contributions to the error (1.8) mainly come from the discretisation of the drift of the position process and of the drift of the velocity process. Each of these components will be separated in the terms before the collision with the reflecting boundary and after the collision. As seen in the sketched proof in Section C.1, the Itô's formula is applied two times in the terms of the decomposition of the error (2.3). Those terms involve the function F in (1.7) which does not have apriori a sufficient regularity. To overcome this difficulty, we first smooth the function F for each variables (t, x, u), with the mollifying sequences $(\beta_k, \rho_l, g_m)_{k,l,m\geq 1}$.

Smooth approximation of F. We construct $(\beta_k)_{k\geq 1}$, $(\rho_l)_{l\geq 1}$ and $(g_m)_{m\geq 1}$, some positive approximations to the identity such that:

$$
\mathrm{Supp}(\beta_k) \subset \left(0, \frac{T}{k}\right), \qquad \mathrm{Supp}(\rho_l) \subset \left(-\frac{1}{l}, 0\right) \qquad \text{and} \qquad \mathrm{Supp}(g_m) = \mathbb{R}. \tag{3.1}
$$

For $(\beta_k)_{k\geq 1}$, we consider the function $t \mapsto \beta(t)$ defined on $\mathbb{R}$ by

$$
\beta(t) = \begin{cases} \exp\left(-\dfrac{1}{t(T-t)}\right) & \text{for } t \in (0,T)\ , \\ 0 \quad \text{otherwise.} & \end{cases} \tag{3.2}
$$

Then for $k \geq 1$, we set $\beta_k(t) = C' k\beta(kt)$ where C' is such that $\int_{[0,T]} \beta(t)\,dt = \frac{1}{C'}$. With the choice for the support of β to be included in $(0,T)$, we have that any convolution on $[0,T]$ is zero at $t = 0$. For example consider the function $h\colon [0,T] \mapsto \mathbb{R}$, then the function $\hat{h}\colon s \mapsto \int_{[0,T]} \beta_k(s-\tau)h(\tau)\,d\tau$ is such that for any $k \geq 1$, $\hat{h}(0) = 0$. We can easily see this in the following graph where we consider $T = 1$, $k = 10$ and $h\colon s \mapsto \mathbb{1}_{[0,1]}(s)(2-s)$.

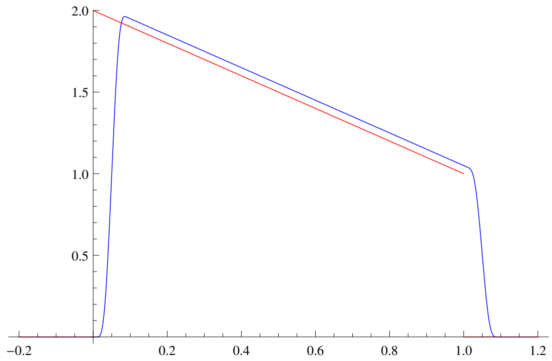

Figure 1: Convolution (in blue) on $[0,T]$ between $s \mapsto h(s) = \mathbb{1}_{[0,1]}(s)(2-s)$ (in red) and mollifier β_k

For $(\rho_l)_{l\geq 1}$, we consider the generating function $x \mapsto \rho(x)$, defined on $\mathbb{R}$ by

$$\rho(x) = \begin{cases} \exp\left(-\dfrac{1}{x(-1-x))}\right) & \text{for } x \in (-1,0)\,, \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.3}$$

Then for any $l \geq 1$, $\rho_l(x) = Cl\rho(lx)$ where C is such that $\displaystyle\int_{\mathbb{R}} \rho(x)\,dx = \frac{1}{C}$.

For the sequence $(g_m)_{m\geq 1}$, we choose to use the Gaussian kernel $u \mapsto g(u)$ on $\mathbb{R}$ with

$$g(u) = \frac{1}{\sqrt{(2\pi)}} \exp\left(-\frac{u^2}{2}\right) \tag{3.4}$$

by taking $g_m(u) = mg(mu)$. We obtain the smooth function: $\forall (t,x,u) \in \overline{Q_T}$,

$$F_{k,l,m}(t,x,u) = \int_{Q_T} F(\tau,y,v)\beta_k(t-\tau)\rho_l(x-y)g_m(u-v)\,d\tau dy dv. \tag{3.5}$$

We mention again that for the choice of the mollifying sequence $(\beta_k)_{k\geq 1}$ to have support on $\left(0, \frac{T}{k}\right)$, we obtain that

$$\forall (x,u) \in \mathcal{D}\times\mathbb{R}, \quad F_{k,l,m}(0,x,u) = 0\,.$$

We denote by L the infinitesimal generator for the process $(X_t,U_t)_{0\leq t\leq T}$:

$$L = u\partial_x + b(x,u)\partial_u + \frac{\sigma^2}{2}\partial^2_{uu}.$$

As a corollary of Lemma 5.2 in Section 5, we have

Corollary 3.1. *The smooth function $F_{k,l,m}$ defined on $\overline{Q_T}$ satisfies the following equality for any (t,x,u) in the interior of Q_T:*

$$\left(\frac{\partial}{\partial t} + L\right) F_{k,l,m}(t,x,u) = R_{k,l,m}[F](t,x,u). \tag{3.6}$$

with

$$R_{k,l,m}[F](t,x,u) = R^{\text{Sp}}_{k,l,m}[F](t,x,u) + R^{\text{Tm}}_{k,l,m}[F](t,x,u)$$

where

$$\begin{aligned} R^{\text{Sp}}_{k,l,m}[F](t,x,u) &:= (\partial_x F * (ug_m\rho_l\beta_k))(t,x,u) + b(x,u)\cdot((\partial_u F * (g_m\rho_l\beta_k))(t,x,u)) \\ &\quad - ((b\cdot\partial_u F) * (g_m\rho_l\beta_k))(t,x,u) \end{aligned}$$

$$R^{\text{Tm}}_{k,l,m}[F](t,x,u) := \beta_k(t)F(0,\cdot,\cdot) * (g_m\rho_l)(x,u).$$

Proof. We apply Lemma 5.2 by noticing that for any $(t,x,u) \in Q_T$, $f(t,x,u) = F(T-t,x,u)$. We have by the definition of $f_{k,l,m}$ in (5.4) for any $(\tau,y,v) \in Q_T$ and since $\widetilde{\beta}_k(t) = \beta_k(-t)$

$$
\begin{aligned}
f_{k,l,m}(T-\tau,y,v) &= \int_{Q_T} f(s,x,u)\widetilde{\beta}_k(T-\tau-s)\rho_l(y-x)g_m(v-u)\,dsdxdu \\
&= \int_{Q_T} f(T-t,x,u)\widetilde{\beta}_k(t-\tau)\rho_l(y-x)g_m(v-u)\,dtdxdu \\
&= \int_{Q_T} F(t,x,u)\beta_k(\tau-t)\rho_l(y-x)g_m(v-u)\,dtdxdu = F_{k,l,m}(\tau,y,v)\,,
\end{aligned}
\tag{3.7}
$$

where the change of variable $s \to T-t$ was performed and we obtain that $\partial_t f_{k,l,m}(T-t,x,y) = -\partial_t F_{k,l,m}(t,x,y)$. Now we consider the rest term $R^{\text{Tm}}_{k,l,m}[f]$ of Lemma 5.2 and have

$$
\begin{aligned}
R^{\text{Tm}}_{k,l,m}[f](T-\tau,y,v) &= \widetilde{\beta}_k((T-\tau)-T)f_{l,m}(T,y,v) \\
&= \widetilde{\beta}_k(-\tau)F_{l,m}(0,y,v) = \beta_k(\tau)F_{l,m}(0,y,v) \\
&= R^{\text{Tm}}_{k,l,m}[F](\tau,y,v)
\end{aligned}
$$

with $F_{l,m}(0,\cdot,\cdot) = F(0,\cdot,\cdot) * (g_m\rho_l)(\cdot,\cdot)$.

From these equalities it is straightforward to conclude the result of the lemma. $\square$

$R^{\text{Sp}}_{k,l,m}$ denotes mainly the spatial contribution to the regularization error. Since we choose ψ in $\mathcal{C}^{1,1}_c(\mathcal{D}\times\mathbb{R};\mathbb{R})$, applying Theorem 2.1, we obtain that $\partial_x F$ and $\partial_u F$ are well defined and belong in $\mathcal{C}([0,T];L^\infty(\mathcal{D}\times\mathbb{R});\mathbb{R}) \cap \mathcal{C}([0,T]\times\overline{D}\times\mathbb{R};\mathbb{R})$. Later in Lemma 5.3 we prove that $R^{\text{Sp}}_{k,l,m}$ converges uniformly to 0 as k, l and m go to infinity.

$R^{\text{Tm}}_{k,l,m}$ is mostly a temporal contribution to the regularization error. We prove that $\int_0^T R^{\text{Tm}}_{k,l,m}[F]$ converges uniformly toward $F(0,\cdot,\cdot)$ as k, l and m go to infinity.

Now we can go back to the error decomposition made in (2.3)

$$
\begin{aligned}
&\mathbb{E}\psi(X^{X_0,U_0}_T, U^{X_0,U_0}_T) - \mathbb{E}\psi(\bar{X}^{X_0,U_0}_T, \bar{U}^{X_0,U_0}_T) \\
&= \mathbb{E}\sum_{i=0}^{n-1}\left(F(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i}) - F(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1}\left(F(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})}) - F(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\right),
\end{aligned}
$$

and introduce the smooth solution and pick k such that $\text{Supp}(\beta_k) \subset (0,\Delta t \wedge \varepsilon_0)$ with ε_0 defined in ($H_{\textit{Weak Error}}$):

$$
\begin{aligned}
&\mathbb{E}\psi(X^{X_0,U_0}_T, U^{X_0,U_0}_T) - \mathbb{E}\psi(\bar{X}^{X_0,U_0}_T, \bar{U}^{X_0,U_0}_T) \\
&= \mathbb{E}\sum_{i=0}^{n-1}\left(F_{k,l,m}(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i}) - F_{k,l,m}(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1}\left(F_{k,l,m}(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})}) - F_{k,l,m}(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1}\left((F-F_{k,l,m})\,(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i})\right) - \mathbb{E}\sum_{i=0}^{n-1}\left((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1}\left((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})})\right) - \mathbb{E}\sum_{i=0}^{n-1}\left((F-F_{k,l,m})\,(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\right).
\end{aligned}
\tag{3.8}
$$

3.1 On the error terms introduced by regularizing the solution

The regularisation in time and space components introduce some errors that we analyse. Special care is taken for the time regularisation since it introduced a term $R^{\text{Tm}}_{k,l,m}$ that cannot be bounded uniformly in k. We denote by $\text{Reg}_{k,l,m}$ the term:

$$\begin{aligned}\text{Reg}_{k,l,m} :=& \mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i})\Big) - \mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\Big)\\ &+\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})})\Big) - \mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\Big)\,.\end{aligned} \tag{3.9}$$

Assuming no collision takes place on the first discretisation interval

If no collision takes place on (t_0,t_1), we have that

$$\begin{aligned}\text{Reg}_{k,l,m} =& \mathbb{E}\sum_{i=1}^{n-1}\Big((F-F_{k,l,m})\,(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i})\Big) - \mathbb{E}\sum_{i=1}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\Big)\\ &+\mathbb{E}\sum_{i=1}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})})\Big) - \mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\Big)\\ &+\mathbb{E}F(0,X_0,U_0)\end{aligned} \tag{3.10}$$

Assuming a collision takes place on the first discretisation interval

If a collision takes place on (t_0,t_1), we have that

$$\begin{aligned}\text{Reg}_{k,l,m} =& \mathbb{E}\sum_{i=1}^{n-1}\Big((F-F_{k,l,m})\,(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i})\Big) - \mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\Big)\\ &+\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})})\Big) - \mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\Big)\\ &+\mathbb{E}F(0,X_0,U_0)\end{aligned} \tag{3.11}$$

In both cases we denote

$$\text{Reg}_{k,l,m} = \epsilon^{\text{Reg}}_{k,l,m} + \mathbb{E}F(0,X_0,U_0)\,. \tag{3.12}$$

For any $i\in\{0,\cdots,n-1\}$, we denote the error obtained before a collision as:

$$\epsilon_{\text{BR}}(i) := \mathbb{E}\left[\Big(F_{k,l,m}(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i}) - F_{k,l,m}(\theta_i,\bar{X}^{X_0,U_0}_{\theta_i},\bar{U}^{X_0,U_0}_{\theta_i^-})\Big)\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\right] \tag{3.13}$$

and after the collision as

$$\epsilon_{\text{AR}}(i) := \mathbb{E}\left[\Big(F_{k,l,m}(\theta_i,\bar{X}^{X_0,U_0}_{\theta_i},\bar{U}^{X_0,U_0}_{\theta_i}) - F_{k,l,m}(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\Big)\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\right]. \tag{3.14}$$

If no collision occurs on (t_i,t_{i+1}) the error is denoted as

$$\epsilon_{\text{NoR}}(i) := \mathbb{E}\left[\Big(F_{k,l,m}(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i}) - F_{k,l,m}(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\Big)\mathbb{1}_{\{\theta_i=t_i\}}\right]. \tag{3.15}$$

The ϵ_{BR}, ϵ_{AR}, ϵ_{NoR} are the terms that we develop through an application of Itô's formula. On each sub-intervals $[t_i, \theta_i))$, we introduce the partial differential operator

$$\mathcal{L}_{\text{BR}} h(t,x,u) := \left(\bar{U}_{t_i} \partial_x h + b(\bar{X}_{t_i}, \bar{U}_{t_i}) \partial_u h + \frac{\sigma^2}{2} \partial^2_{uu} h \right)(t,x,u),$$

and on the interval $[\theta_i, t_{i+1})$ we define:

$$\mathcal{L}_{\text{AR}} h(t,x,u) := \left(-\bar{U}_{t_i} \partial_x h + b(\bar{X}_{\nu(t)}, \bar{U}_{\nu(t)}) \partial_u h + \frac{\sigma^2}{2} \partial^2_{uu} h \right)(t,x,u),$$

if no collision occurs on (t_i, t_{i+1}), for any $h \in C^{1,1,2}(Q_t)$, we have the operator

$$\mathcal{L}_{\text{NoR}} h(t,x,u) := \left(\bar{U}_{t_i} \partial_x h + b(\bar{X}_{t_i}, \bar{U}_{t_i}) \partial_u h + \frac{\sigma^2}{2} \partial^2_{uu} h \right)(t,x,u),$$

where $h \in C^{1,1,2}(Q_t)$. The subscript BR signifies "before reflection", AR signifies "after reflection" and NoR signifies "no reflection". The $\text{sign}(\bar{Y}_t)$ dependency is in fact a constant term such that

$$\text{sign}(\bar{Y}_t) = \begin{cases} 1, & \forall t \in [t_i, \theta_i),\ \theta_i \neq t_i, \quad \text{BR} \\ -1, & \forall t \in [\theta_i, t_{i+1}),\ \theta_i \neq t_i, \quad \text{AR} \\ 1, & \forall t \in [t_i, t_{i+1}),\ \theta_i = t_i. \quad \text{NoR} \end{cases}$$

or to be more explicit, $\text{sign}(\bar{Y}_t)$ equals 1 in $\mathcal{L}_{\text{BR}}$ and $\mathcal{L}_{\text{NoR}}$ and -1 for $\mathcal{L}_{\text{AR}}$. It can be seen that the differential operator $\mathcal{L}_{\text{BR}}$ and $\mathcal{L}_{\text{NoR}}$ (before a collision or if no collision occurs) are similar, so the results from one apply to the other if the time interval of application is adjusted accordingly.

By applying the Itô formula to the first two terms of (3.8), we obtain that:

$$\begin{aligned}
&\mathbb{E}\psi(X_T^{X_0,U_0}, U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0}, \bar{U}_T^{X_0,U_0}) \\
&= \mathbb{E} \sum_{i=0}^{n-1} \left(F_{k,l,m}(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0}) - F_{k,l,m}(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)^-}^{X_0,U_0}) \right) \\
&\quad + \mathbb{E} \sum_{i=0}^{n-1} \left(F_{k,l,m}(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0}) - F_{k,l,m}(t_{i+1}^-, \bar{X}_{t_{i+1}^-}^{X_0,U_0}, \bar{U}_{t_{i+1}^-}^{X_0,U_0}) \right) + \text{Reg}_{k,l,m} \\
&= \sum_{i=0}^{n-1} (\epsilon_{\text{BR}}(i) + \epsilon_{\text{AR}}(i) + \epsilon_{\text{NoR}}(i)) + \epsilon_{k,l,m}^{\text{Reg}} + \mathbb{E}F(0, X_0, U_0) \\
&= -\sum_{i=0}^{n-1} \mathbb{E}\left[\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\partial_t + \mathcal{L}_{\text{BR}}) F_{k,l,m}(s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0}) ds \right] \\
&\quad - \sum_{i=0}^{n-1} \mathbb{E}\left[\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} (\partial_t + \mathcal{L}_{\text{AR}}) F_{k,l,m}(s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds \right] \\
&\quad - \sum_{i=0}^{n-1} \mathbb{E}\left[\mathbb{1}_{\{\theta_i = t_i\}} \int_{t_i}^{t_{i+1}} (\partial_t + \mathcal{L}_{\text{NoR}}) F_{k,l,m}(s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds \right] + \epsilon_{k,l,m}^{\text{Reg}} + \mathbb{E}F(0, X_0, U_0).
\end{aligned} \tag{3.16}$$

The stochastic integrals terms are actually martingales since by Theorem 2.1, $\partial_u F \in L^\infty(Q_T)$. Since $F_{k,l,m}$ is a solution to the equation (3.6):

$$
\begin{aligned}
&\mathbb{E}\psi(X_T^{X_0,U_0},U_T^{X_0,U_0})-\mathbb{E}\psi(\bar{X}_T^{X_0,U_0},\bar{U}_T^{X_0,U_0})\\
&=\sum_{i=0}^{n-1}\left(\epsilon_{\mathrm{BR}}(i)+\epsilon_{\mathrm{AR}}(i)+\epsilon_{\mathrm{NoR}}(i)\right)+\epsilon_{k,l,m}^{\mathrm{Reg}}+\mathbb{E}F(0,X_0,U_0)\\
&=\sum_{i=0}^{n-1}\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}(L-\mathcal{L}_{\mathrm{BR}})F_{k,l,m}(s,\bar{X}_s^{X_0,U_0},\bar{U}_s^{X_0,U_0})\,ds\\
&\quad+\sum_{i=0}^{n-1}\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(L-\mathcal{L}_{\mathrm{AR}})F_{k,l,m}(s,\bar{X}_s^{X_0,U_0},\bar{U}_s^{X_0,U_0})\,ds\\
&\quad+\sum_{i=0}^{n-1}\mathbb{E}\mathbb{1}_{\{\theta_i=t_i\}}\int_{t_i}^{t_{i+1}}(L-\mathcal{L}_{\mathrm{NoR}})F_{k,l,m}(s,\bar{X}_s^{X_0,U_0},\bar{U}_s^{X_0,U_0})\,ds\\
&\quad-\sum_{i=0}^{n-1}\mathbb{E}\int_{t_i}^{t_{i+1}}R_{k,l,m}^{\mathrm{Sp}}[F](s,\bar{X}_s^{X_0,U_0},\bar{U}_s^{X_0,U_0})\,ds\\
&\quad-\sum_{i=0}^{n-1}\mathbb{E}\int_{t_i}^{t_{i+1}}R_{k,l,m}^{\mathrm{Tm}}[F](s,\bar{X}_s^{X_0,U_0},\bar{U}_s^{X_0,U_0})\,ds+\mathbb{E}F(0,X_0,U_0)+\epsilon_{k,l,m}^{\mathrm{Reg}}\,.
\end{aligned}
\tag{3.17}
$$

Remark 3.2. *By Theorem 2.1, F is in $W^{(1,1),2}(Q_T)$. A generalized Ito's Lemma (see e.g. Theorem 1, page 122 of [18]) with the extension for unbounded domains and hypo-elliptic diffusions, should have been applied in this part of the proof, instead of regularising F.*

We now present a lemma that gives the convergence of the various terms that compose the error obtained by regularization.

Lemma 3.3. *We have that*

(i) $\left|\epsilon_{k,l,m}^{Reg}\right| \xrightarrow{k,l,m\to\infty} 0$

(ii) $\left|\sum_{i=0}^{n-1}\mathbb{E}\int_{t_i}^{t_{i+1}}R_{k,l,m}^{Tm}[F](s,\bar{X}_s^{X_0,U_0},\bar{U}_s^{X_0,U_0})\,ds-EF(0,X_0,U_0)\right| \xrightarrow{k,l,m\to\infty} 0$

Proof. **Convergence *(i)*.**

According to the Lemma 4.5, F is continuous and bounded on $\overline{Q_T}$ and in fact we can extend naturally F as a continuous, bounded function on $[0,T]\times\mathbb{R}\times\mathbb{R}$ (for an example of such an extension on the whole domain see the calculations (4.25) in Section 4 and take $F(t,x,u)=f(T-t,x,u)$).

We recall that if a collision occurs on the first interval (t_0,t_1), that we have that

$$
\begin{aligned}
\epsilon_{k,l,m}^{\mathrm{Reg}}&=\mathbb{E}\sum_{i=1}^{n-1}\Big((F-F_{k,l,m})\,(t_i,\bar{X}_{t_i}^{X_0,U_0},\bar{U}_{t_i}^{X_0,U_0})\Big)-\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t_{i+1}^-),\bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0},\bar{U}_{\nu(t_{i+1}^-)-}^{X_0,U_0})\Big)\\
&\quad+\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t_{i+1}^-),\bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0},\bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0})\Big)-\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(t_{i+1}^-,\bar{X}_{t_{i+1}^-}^{X_0,U_0},\bar{U}_{t_{i+1}^-}^{X_0,U_0})\Big)
\end{aligned}
\tag{3.18}
$$

and we apply Lemma A.6, in the Appendix section A, which states we have that $F_{k,l,m}$ converges uniformly on any compact of $(0,T]\times\mathbb{R}\times\mathbb{R}$. In our case, we consider the compact $[\varepsilon_0\wedge t_1,T]\times\mathbb{R}\times\mathbb{R}$.

By condition ($H_{Weak\ Error}$) (see Remark 1.5), the first collision time $\nu(t_1^-)$ is such that $\nu(t_1^-)\geq\varepsilon_0$, therefore the random variables $F_{k,l,m}(t_i,\bar{X}_{t_i}^{X_0,U_0},\bar{U}_{t_i}^{X_0,U_0})$ (this term considered only for $i\geq 1$), $F_{k,l,m}(\nu(t_{i+1}^-),\bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0},\bar{U}_{\nu(t_{i+1}^-)-}^{X_0,U_0})$, $F_{k,l,m}(\nu(t_{i+1}^-),\bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0},\bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0})$ and $F_{k,l,m}(t_{i+1}^-,\bar{X}_{t_{i+1}^-}^{X_0,U_0},\bar{U}_{t_{i+1}^-}^{X_0,U_0})$ converge almost surely to $F(t_i,\bar{X}_{t_i}^{X_0,U_0},\bar{U}_{t_i}^{X_0,U_0})$, $F(\nu(t_{i+1}^-),\bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0},\bar{U}_{\nu(t_{i+1}^-)-}^{X_0,U_0})$, $F(\nu(t_{i+1}^-),\bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0},\bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0})$ and, respectively, $F(t_{i+1}^-,\bar{X}_{t_{i+1}^-}^{X_0,U_0},\bar{U}_{t_{i+1}^-}^{X_0,U_0})$. Since F is a bounded function, then $\epsilon_{k,l,m}^{\mathrm{Reg}}$ goes to zero as k,l,m go to infinity by the Dominated Convergence Theorem.

Similar arguments apply if there is no collision on the first interval (t_0,t_1).

Convergence *(ii)*.

Since k has been chosen such that $\mathrm{Supp}(\beta_k) \subset (0, \Delta t \wedge \varepsilon_0)$, and since no collision occurs on $(0, \Delta t \wedge \varepsilon_0) \equiv (t_0, t_1 \wedge \varepsilon_0)$ (see Remark 1.5) then we have that

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} R^{\mathrm{Tm}}_{k,l,m}[F](s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds = \int_0^T R^{\mathrm{Tm}}_{k,l,m}[F](s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds$$
$$= \int_0^T \beta_k(s) F(0,\cdot,\cdot) * (\rho_l g_m)(\bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds$$
$$= \int_0^{\Delta t \wedge \varepsilon_0} \beta_k(s) F(0,\cdot,\cdot) * (\rho_l g_m)(X_0 + sU_0, U_0)\, ds\,.$$

By uniform convergence arguments of convolutions used in the previous section we have that $F(0,\cdot,\cdot) * (\rho_l g_m)(X_0 + sU_0, U_0)$ converges a.s. to $F(0, X_0 + sU_0, U_0)$. We introduce the function $g\colon [0,T] \mapsto \mathbb{R}$, such that for any $s \in [0,T]$ $g(s) = F(0, X_0 + sU_0, U_0)$. By Lemma 4.5, we have that F is continuous on Q_T, therefore g is a continuous function on $[0,T]$.

For any $\epsilon > 0$, there exists $\delta > 0$ such that $|g(0) - g(s)| < \varepsilon$, for any $s \in (0, \delta)$.

We recall that $\mathrm{Supp}(\beta_k) \in (0, \frac{T}{k})$ so the previous equality becomes

$$\int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s) F(0, X_0 + sU_0, U_0)\, ds - F(0, X_0, U_0) = \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s) g(s)\, ds - g(0)$$
$$= \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s)(g(s) - g(0))\, ds$$

so for every k such that $\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0 \le \delta$, we obtain that

$$\left| \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s)(g(s) - g(0))\, ds \right| \le \varepsilon \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s)\, ds = \varepsilon\,.$$

Thus, $\int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s) F(0, X_0 + sU_0, U_0)\, ds$ converges almost surely towards $F(0, X_0, U_0)$. As F is a bounded function therefore

$$\left| \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s) F(0, X_0 + sU_0, U_0)\, ds \right| \le \|F\|_{L^\infty(Q_T)} \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s)\, ds = \|F\|_{L^\infty(Q_T)}$$

then by the dominated convergence theorem, we obtain the desired result. □

In order to simplify the writing, we remove the references to the initial conditions and write simply $(\bar{X}_t, \bar{U}_t)$ as $(\bar{X}_t^{X_0,U_0}, \bar{U}_t^{X_0,U_0})$.

For all $i \in \{0, \ldots, n-1\}$, according to the definition of $\mathcal{L}_{\mathrm{BR}}$, the term under the first summation in the r.h.s of equality (3.17) is rewritten as:

$$\begin{aligned}
&\mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (L - \mathcal{L}_{\mathrm{BR}}) F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&= \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\bar{U}_s - \bar{U}_{\eta(s)}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&\quad + \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} \left(b(\bar{X}_s, \bar{U}_s) - b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)}) \right) \partial_u F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&=: \epsilon_{\mathrm{BR}}^{\bar{X}}(i) + \epsilon_{\mathrm{BR}}^{\bar{U}}(i).
\end{aligned} \tag{3.19}$$

The third sum in the r.h.s. of equality (3.17) corresponds to the case without reflection, and it can be developed similarly to

The term under the second summation in the r.h.s. of equality (3.17) is:

$$
\begin{aligned}
&\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(L-\mathcal{L}_{\text{AR}})F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\\
&=\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(\bar{U}_s-\bar{U}_{\eta(s)}\operatorname{sign}(\bar{Y}_s))\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)ds\\
&\quad+\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\left(b(\bar{X}_s,\bar{U}_s)-b(\bar{X}_{\nu(s)},\bar{U}_{\nu(s)})\right)\partial_u F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\\
&=\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(\bar{U}_s+\bar{U}_{\eta(s)})\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\\
&\quad+\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\left(b(\bar{X}_s,\bar{U}_s)-b(\bar{X}_{\nu(s)},\bar{U}_{\nu(s)})\right)\partial_u F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\\
&=:\epsilon^{\bar{X}}_{\text{AR}}(i)+\epsilon^{\bar{U}}_{\text{AR}}(i).
\end{aligned}
\tag{3.20}
$$

We recall that for $s\in[\theta_i,t_{i+1})$, $\bar{U}_s$ is the velocity after specular reflection, so there is a change of sign at θ_i.

The error is then further decomposed with contribution from the discretization of the drift of the position process $(\bar{X}_t)_{0\le t\le T}$ and a contribution from the drift of the velocity process $(\bar{U}_t)_{0\le t\le T}$. We denote these errors before the reflection as $\epsilon^{\bar{X}}_{\text{BR}}(i)$, $\epsilon^{\bar{U}}_{\text{BR}}(i)$ respectively, after the reflection $\epsilon^{\bar{X}}_{\text{AR}}(i)$ and $\epsilon^{\bar{U}}_{\text{AR}}(i)$. We finally denote $\epsilon^{\bar{X}}_{\text{NoR}}(i)$ and $\epsilon^{\bar{U}}_{\text{NoR}}(i)$ the error obtained when no reflection occurs on the interval. The superscript $^{\bar{X}}$ denotes the error related to the approximation of the position of the particle while the superscript $^{\bar{U}}$ denotes the error due to the approximation of the velocity of the particle.

3.2 Contribution to the error $\epsilon^{\bar{X}}$ of the discretized drift on the position process

Contribution to the error before the reflection

We begin by developing the error produced by the discretization of position process, before reflection:

$$
\begin{aligned}
\epsilon^{\bar{X}}_{\text{BR}}(i)&=\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}(\bar{U}_s-\bar{U}_{\eta(s)}))\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\\
&=\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}(s-t_i)b(\bar{X}_{t_i},\bar{U}_{t_i})\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\\
&\quad+\sigma\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}(W_s-W_{t_i})\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds.
\end{aligned}
\tag{3.21}
$$

We consider the inner integral

$$
\begin{aligned}
&\int_{t_i}^{\theta_i}(\bar{U}_s-\bar{U}_{\eta(s)}))\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\\
&=\int_{t_i}^{\theta_i}(s-t_i)b(\bar{X}_{t_i},\bar{U}_{t_i})\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds+\int_{t_i}^{\theta_i}(W_s-W_{t_i})\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds.
\end{aligned}
$$

The second term of this equality is treated separately by conditioning w.r.t $\mathcal{F}_{t_i}$. For any $s\ge t_i$, the increment $W_s-W_{t_i}$ is independent to the $\sigma-$algebra $\mathcal{F}_{t_i}$, so by introducing the probability density function of the standard Gaussian random variable denoted $p_{\mathcal{N}(0,1)}$, we have:

$$
\begin{aligned}
&\mathbb{E}\left[\int_{t_i}^{\theta_i}(W_s-W_{t_i})\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\Big|\mathcal{F}_{t_i}\right]\\
&=\mathbb{E}\left[\int_{t_i}^{\theta_i}(W_s-W_{t_i})\partial_x F_{k,l,m}(s,\bar{X}_{t_i}+(s-t_i)\bar{U}_{t_i},\bar{U}_{t_i}+b(\bar{X}_{t_i},\bar{U}_{t_i})(s-t_i)+\sigma(W_s-W_{t_i}))\,ds\Big|\mathcal{F}_{t_i}\right]\\
&=\int_{t_i}^{\theta_i}\sqrt{s-t_i}\,ds\int_{\mathbb{R}}w\partial_x F_{k,l,m}(s,\bar{X}_{t_i}+(s-t_i)\bar{U}_{t_i},\bar{U}_{t_i}+b(\bar{X}_{t_i},\bar{U}_{t_i})(s-t_i)+\sigma\sqrt{s-t_i}w)p_{\mathcal{N}(0,1)}(w)dw.
\end{aligned}
$$

The integral can be transformed to obtain a derivative of the Gaussian density:

$$
\begin{aligned}
&\int_{\mathbb{R}} w\partial_x F_{k,l,m}(s, \bar{X}_{t_i} + (s-t_i)\bar{U}_{t_i}, \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(s-t_i) + \sigma\sqrt{s-t_i}w) p_{\mathcal{N}(0,1}(w)\, dw \\
&= -\int_{\mathbb{R}} \partial_x F_{k,l,m}(s, \bar{X}_{t_i} + (s-t_i)\bar{U}_{t_i}, \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(s-t_i) + \sigma\sqrt{s-t_i}w) \frac{d}{dw} p_{\mathcal{N}(0,1)}(w)\, dw \\
&= \sigma\sqrt{s-t_i} \int_{\mathbb{R}} \frac{\partial^2 F_{k,l,m}}{\partial u \partial x}(s, \bar{X}_{t_i} + (s-t_i)\bar{U}_{t_i}, \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(s-t_i) + \sigma\sqrt{s-t_i}w) p_{\mathcal{N}(0,1)}(w)\, dw
\end{aligned}
$$

The last equality is obtained from an integration by parts. By Lemma 4.6, we have that $\partial_x F$ is a bounded function, thus $\partial_x F_{k,l,m}$ is also bounded, and as $p_{\mathcal{N}(0,1)}(w) \to 0$ as $|w| \to \infty$, the boundary terms from the i.b.p. are 0.

We can rewrite:

$$
\mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (W_s - W_{t_i}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds = \sigma^2 \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (s-t_i) \frac{\partial^2 F_{k,l,m}}{\partial u \partial x}(s, \bar{X}_s, \bar{U}_s)\, ds.
$$

Finally, we obtain that:

$$
\epsilon^{\bar{X}}_{\mathrm{BR}}(i) = \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (s-t_i) \left(b(\bar{X}_{t_i}, \bar{U}_{t_i}) \frac{\partial F_{k,l,m}}{\partial x}(s, \bar{X}_s, \bar{U}_s) + \sigma^2 \frac{\partial^2 F_{k,l,m}}{\partial u \partial x}(s, \bar{X}_s, \bar{U}_s) \right) ds. \tag{3.22}
$$

The $(s - t_i)$ factor in the integral allows us to obtain the linear decrease of the error in Δt, so we express all the other error terms in this form. Similar calculations give:

$$
\epsilon^{\bar{X}}_{\mathrm{NoR}}(i) = \mathbb{E}\mathbb{1}_{\{\theta_i = t_i\}} \int_{t_i}^{t_{i+1}} (s-t_i) \left(b(\bar{X}_{t_i}, \bar{U}_{t_i}) \frac{\partial F_{k,l,m}}{\partial x}(s, \bar{X}_s, \bar{U}_s) + \sigma^2 \frac{\partial^2 F_{k,l,m}}{\partial u \partial x}(s, \bar{X}_s, \bar{U}_s) \right) ds. \tag{3.23}
$$

Contribution to the error after the reflection

We analyze now the contribution to the error produced by the discretisation of the drift in the position process, after reflection on any interval $[\theta_i, t_{i+1}]$, given by:

$$
\begin{aligned}
\epsilon^{\bar{X}}_{\mathrm{AR}}(i) &= \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} (\bar{U}_s + \bar{U}_{\eta(s)}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&= \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} \Big[- b(\bar{X}_{t_i}, \bar{U}_{t_i})(\theta_i - t_i) - \sigma(W_{\theta_i} - W_{t_i}) \\
&\qquad\qquad + b(\bar{X}_{\theta_i}, \bar{U}_{\theta_i})(s - \theta_i) + \sigma(W_s - W_{\theta_i}) \Big] \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds.
\end{aligned} \tag{3.24}
$$

The terms that involve Brownian increments are analysed separately starting with the increment before the jump, in the same way as the previous paragraph, in order to obtain a term of the type $wp_{\mathcal{N}(0,1)}(w)$:

$$
\begin{aligned}
&\mathbb{E}\left[\int_{\theta_i}^{t_{i+1}} \sigma(W_{\theta_i} - W_{t_i}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \Big| \mathcal{F}_{t_i} \right] \\
&= \sigma \mathbb{E}\left[(W_{\theta_i} - W_{t_i}) \int_{\theta_i}^{t_{i+1}} \partial_x F_{k,l,m}(s, -(s-\theta_i)\bar{U}_{t_i}, \bar{U}_{\theta_i} + b(0, \bar{U}_{\theta_i})(s-\theta_i) + \sigma(W_s - W_{\theta_i}))\, ds \Big| \mathcal{F}_{t_i} \right] \\
&= \sigma \mathbb{E}\left[(W_{\theta_i} - W_{t_i}) \int_{\theta_i}^{t_{i+1}} \mathbb{E}\big[\partial_x F_{k,l,m}(s, -(s-\theta_i)\bar{U}_{t_i}, \bar{U}_{\theta_i} + b(0, \bar{U}_{\theta_i})(s-\theta_i) + \sigma(W_s - W_{\theta_i})) \mid \mathcal{F}_{\theta_i} \big] ds \Big| \mathcal{F}_{t_i} \right].
\end{aligned}
$$

In order to simplify notations, we introduce the function $I \colon \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}$ such that:

$$
I(u, \theta_i, s) = \mathbb{E}\big[\partial_x F_{k,l,m}(s, -(s-\theta_i)\bar{U}_{t_i}, u + b(0,u)(s-\theta_i) + \sigma(W_s - W_{\theta_i})) \mid \mathcal{F}_{\theta_i} \big].
$$

The previous equality then becomes:

$$\begin{aligned}
&\mathbb{E}\left[\int_{\theta_i}^{t_{i+1}} \sigma(W_{\theta_i} - W_{t_i})\partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \sigma\mathbb{E}\left[(W_{\theta_i} - W_{t_i}) \int_{\theta_i}^{t_{i+1}} I(\bar{U}_{\theta_i}, \theta_i, s)\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \sigma\mathbb{E}\left[(W_{\theta_i} - W_{t_i}) \int_{\theta_i}^{t_{i+1}} I(-\bar{U}_{t_i} - b(\bar{X}_{t_i}, \bar{U}_{t_i})(\theta_i - t_i) - \sigma(W_{\theta_i} - W_{t_i}), \theta_i, s)\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \sigma\mathbb{E}\left[\sqrt{\theta_i - t_i} \int_{\theta_i}^{t_{i+1}} ds \int_{\mathbb{R}} wI(-\bar{U}_{t_i} - b(\bar{X}_{t_i}, \bar{U}_{t_i})(\theta_i - t_i) - \sigma\sqrt{\theta_i - t_i}w, \theta_i, s) p_{\mathcal{N}(0,1)}(w)\, dw \Big| \mathcal{F}_{t_i}\right],
\end{aligned}$$

and just as before, we can perform an integration by parts with $wp_{\mathcal{N}(0,1)}(w) = -p'_{\mathcal{N}(0,1)}(w)$ to obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_{\theta_i}^{t_{i+1}} \sigma(W_{\theta_i} - W_{t_i})\partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \sigma^2\mathbb{E}\left[\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}}(\theta_i - t_i) \int_{\theta_i}^{t_{i+1}} \int_{\mathbb{R}} \frac{\partial I}{\partial u}(\bar{U}_{\theta_i}, \theta_i, s) p_{\mathcal{N}(0,1)}(w)\, dw\right]
\end{aligned}$$

where:

$$\frac{\partial I}{\partial u}(u, \theta_i, s) = \mathbb{E}\left[\left(1 + (s - \theta_i)\frac{\partial b}{\partial u}(0, u)\right) \frac{\partial^2 F_{k,l,m}}{\partial x \partial u}(s, -(s - \theta_i)\bar{U}_{t_i}, u + b(0, u)(s - \theta_i) + \sigma(W_s - W_{\theta_i})) \mid \mathcal{F}_{\theta_i}\right]$$

so by combining the different results:

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} \sigma(W_{\theta_i} - W_{t_i})\partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds\right] \\
&= -\sigma^2\mathbb{E}\left[\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}}(\theta_i - t_i) \int_{\theta_i}^{t_{i+1}} \left(1 + (s - \theta_i)\frac{\partial b}{\partial u}(0, \bar{U}_{\theta_i})\right) \frac{\partial^2 F}{\partial x \partial u}(s, \bar{X}_s, \bar{U}_s)\, ds\right].
\end{aligned}$$

We now consider the case of the Brownian increment after the jump $(W_s - W_{\theta_i})$, which is independent from $\mathcal{F}_{\theta_i}$, so the calculations will be similar to those for $\epsilon_{\text{BR}}^{\bar{X}}(i)$:

$$\begin{aligned}
&\mathbb{E}\left[\int_{\theta_i}^{t_{i+1}} \sigma(W_s - W_{\theta_i})\partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \Big| \mathcal{F}_{\theta_i}\right] \\
&= \sigma \int_{\theta_i}^{t_{i+1}} \mathbb{E}\left[(W_s - W_{\theta_i})\partial_x F_{k,l,m}(s, -(s-\theta_i)\bar{U}_{t_i}, \bar{U}_{\theta_i} + b(0, \bar{U}_{\theta_i})(s-\theta_i) + \sigma(W_s - W_{\theta_i})) \mid \mathcal{F}_{\theta_i}\right] ds \\
&= \sigma \int_{\theta_i}^{t_{i+1}} \sqrt{s - \theta_i} ds \int_{\mathbb{R}} w\partial_x F_{k,l,m}(s, -(s-\theta_i)\bar{U}_{t_i}, \bar{U}_{\theta_i} + b(0, \bar{U}_{\theta_i})(s-\theta_i) + \sigma\sqrt{s-\theta_i}w) p_{\mathcal{N}(0,1)}(w) dw,
\end{aligned}$$

and after applying once more an i.b.p. (with null boundary terms since $\partial_x F_{k,l,m}$ is bounded and as $|u| \to +\infty$, $p_{\mathcal{N}(0,1)}(w) \to 0$) we obtain:

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} \sigma(W_s - W_{\theta_i})\partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s) ds\right] \\
&= \sigma^2\mathbb{E}\left[\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} (s - \theta_i)\frac{\partial^2 F_{k,l,m}}{\partial u \partial x}(s, \bar{X}_s, \bar{U}_s) ds\right].
\end{aligned}$$

And by combining all these terms in (3.24), we obtain:

$$
\begin{aligned}
\epsilon^{\bar{X}}_{\text{AR}}(i) &= \mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}} (\bar{U}_{\eta(s)}+\bar{U}_s)\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds \\
&= -\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}(\theta_i-t_i)\int_{\theta_i}^{t_{i+1}} b(\bar{X}_{t_i},\bar{U}_{t_i})\frac{\partial F_{k,l,m}}{\partial x}(s,\bar{X}_s,\bar{U}_s)\,ds \\
&\quad+\sigma^2\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}(\theta_i-t_i)\int_{\theta_i}^{t_{i+1}}\left(1+(s-\theta_i)\frac{\partial b}{\partial u}(0,\bar{U}_{\theta_i})\right)\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}(s,\bar{X}_s,\bar{U}_s)\,ds \\
&\quad-\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}} b(\bar{X}_{\theta_i},\bar{U}_{\theta_i})(s-\theta_i)\frac{\partial F_{k,l,m}}{\partial x}(s,\bar{X}_s,\bar{U}_s)\,ds \\
&\quad+\sigma^2\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(s-\theta_i)\frac{\partial^2 F_{k,l,m}}{\partial u\partial x}(s,\bar{X}_s,\bar{U}_s)\,ds.
\end{aligned}
\tag{3.25}
$$

Bounding the errors $\epsilon^{\bar{X}}$ of the position component

By using ($H_{Langevin}$)-(ii), we bound in (3.22):

$$
\begin{aligned}
|\epsilon^{\bar{X}}_{\text{BR}}(i)| &\le \left|\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}(s-t_i)b(\bar{X}_{t_i},\bar{U}_{t_i})\frac{\partial F_{k,l,m}}{\partial x}(s,\bar{X}_s,\bar{U}_s)\,ds\right| \\
&\quad+\left|\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}\sigma^2(s-t_i)\frac{\partial^2 F_{k,l,m}}{\partial u\partial x}(s,\bar{X}_s,\bar{U}_s)\,ds\right| \\
&\le \Delta t\max\{\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})},\sigma^2\}\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}\left(\left|\frac{\partial F_{k,l,m}}{\partial x}(s,\bar{X}_s,\bar{U}_s)\right|+\left|\frac{\partial^2 F_{k,l,m}}{\partial u\partial x}(s,\bar{X}_s,\bar{U}_s)\right|\right)ds \\
&\le \Delta t\max\{\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})},\sigma^2\}\mathbb{E}\int_{t_i}^{t_{i+1}}\left(\left|\frac{\partial F_{k,l,m}}{\partial x}(s,\bar{X}_s,\bar{U}_s)\right|+\left|\frac{\partial^2 F_{k,l,m}}{\partial u\partial x}(s,\bar{X}_s,\bar{U}_s)\right|\right)ds.
\end{aligned}
\tag{3.26}
$$

By (3.23), $\epsilon^{\bar{X}}_{\text{NoR}}(i)$ is bounded by the same term. And in (3.25), by (H_{PDE})-(i):

$$
\begin{aligned}
|\epsilon^{\bar{X}}_{\text{AR}}(i)| &\le \Delta t\,\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})}\,\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\left|\frac{\partial F_{k,l,m}}{\partial x}\right|(s,\bar{X}_s,\bar{U}_s)ds \\
&\quad+\Delta t\sigma^2\left(1+\Delta t\left\|\frac{\partial b}{\partial u}\right\|_{L^\infty(\mathcal{D}\times\mathbb{R})}\right)\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\left|\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}\right|(s,\bar{X}_s,\bar{U}_s)ds \\
&\quad+\Delta t\,\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})}\,\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\left|\frac{\partial F_{k,l,m}}{\partial x}\right|(s,\bar{X}_s,\bar{U}_s)ds \\
&\quad+\Delta t\sigma^2\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\left|\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}\right|(s,\bar{X}_s,\bar{U}_s)ds \\
&\le \Delta t\left(C_b\mathbb{E}\int_{t_i}^{t_{i+1}}\left|\frac{\partial F_{k,l,m}}{\partial x}\right|(s,\bar{X}_s,\bar{U}_s)\,ds+\Delta t C_{\partial_u b,\sigma,T}\mathbb{E}\int_{t_i}^{t_{i+1}}\left|\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}\right|(s,\bar{X}_s,\bar{U}_s)ds\right)
\end{aligned}
\tag{3.27}
$$

where C_b is a constant that only depends on b and $C_{\partial_u b,\sigma,T}$ depends only on $\partial_u b$, σ, and T.

Combining these results and summing from $i=0$ to $i=N-1$, we obtain:

$$
\begin{aligned}
&\sum_{i=0}^{N-1}\left(|\epsilon^{\bar{X}}_{\text{BR}}(i)|+|\epsilon^{\bar{X}}_{\text{AR}}(i)|+|\epsilon^{\bar{X}}_{\text{NoR}}(i)|\right) \\
&\le \Delta t\times C_{b,\partial_u b,\sigma,T}\left(\mathbb{E}\int_0^T\left|\frac{\partial F_{k,l,m}}{\partial x}\right|(s,\bar{X}_s,\bar{U}_s)\,ds+\mathbb{E}\int_0^T\left|\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}\right|(s,\bar{X}_s,\bar{U}_s)\,ds\right).
\end{aligned}
\tag{3.28}
$$

3.3 Analysis of the contribution to the error of the discretized drift on the velocity process

Error contribution before the reflection

We now consider the second term of (3.19), which represents the error introduced by the discretization of the drift of the velocity before the jump. Since $F_{k,l,m}$ is a smooth function, we apply Ito's formula. For the term corresponding to the contribution before the jump, we have:

$$\begin{aligned}\epsilon^{\bar{U}}_{\mathrm{BR}}(i) &= \mathbb{E}\left[\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}\left(b(\bar{X}_s,\bar{U}_s)-b(\bar{X}_{\nu(s)},\bar{U}_{\nu(s)})\right)\partial_u F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)ds\right]\\ &= \mathbb{E}\left[\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i} ds\int_{t_i}^{s}\left(\frac{\partial}{\partial t}+\mathcal{L}_{\mathrm{BR}}\right)\left(\left(b(\bar{X}_q,\bar{U}_q)-b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\right)\partial_u F_{k,l,m}(q,\bar{X}_q,\bar{U}_q)\right)dq\right].\end{aligned}$$

The local martingale that results from the application of Ito's formula is actually a true martingale by considering (H_{PDE})-(i) which gives that the drift b and its derivatives are uniformly bounded and $\partial_u F_{k,l,m}, \partial^2_{uu}F_{k,l,m} \in L^\infty(Q_T)$, for fixed $(k,l,m)\in\mathbb{N}^3$. By the definition for $F_{k,l,m}$ in (3.5) and the fact that $F\in L^\infty(Q_T)$we have that for any $(t,x,u)\in Q_T$

$$\left|\partial^2_{uu}F_{k,l,m}(t,x,u)\right| \le \|F\|_{L^\infty(Q_T)}\int_{Q_T}\beta_k(t-\tau)\rho_l(x-y)\left|\partial^2_{uu}g_m(u-v)\right| d\tau dy dv \le m^2\|F\|_{L^\infty(Q_T)}.$$

Similarly, we show that $\partial_u F_{k,l,m}$ is also bounded for fixed $(k,l,m)\in\mathbb{N}^3$.

We distribute the linear differential operator (recall that the drift b does not depend on time) in the inner integral:

$$\begin{aligned}I_s &:= \int_{t_i}^{s}\left(\frac{\partial}{\partial t}+\mathcal{L}_{\mathrm{BR}}\right)\left(\left(b(\bar{X}_q,\bar{U}_q)-b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\right)\partial_u F_{k,l,m}(q,\bar{X}_q,\bar{U}_q)\right)dq\\ &= \int_{t_i}^{s}\left(b(\bar{X}_q,\bar{U}_q)-b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\right)\frac{\partial}{\partial u}\frac{\partial}{\partial t}F_{k,l,m}(q,\bar{X}_q,\bar{U}_q)\,dq\\ &\quad+\int_{t_i}^{s}\mathcal{L}_{\mathrm{BR}}\left(\left(b(\bar{X}_q,\bar{U}_q)-b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\right)\frac{\partial}{\partial u}F_{k,l,m}(q,\bar{X}_q,\bar{U}_q)\right)dq\\ &= -\int_{t_i}^{s}\left(b(\bar{X}_q,\bar{U}_q)-b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\right)\frac{\partial}{\partial u}LF_{k,l,m}(q,\bar{X}_q,\bar{U}_q)\,dq\\ &\quad+\int_{t_i}^{s}\left(b(\bar{X}_q,\bar{U}_q)-b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\right)\partial_u R_{k,l,m}(q,\bar{X}_q,\bar{U}_q)\,dq\\ &\quad+\int_{t_i}^{s}\mathcal{L}_{\mathrm{BR}}\left(\left(b(\bar{X}_q,\bar{U}_q)-b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\right)\frac{\partial}{\partial u}F_{k,l,m}(q,\bar{X}_q,\bar{U}_q)\right)dq\end{aligned}$$

where we have used the fact that $\partial_t\partial_u F_{k,l,m} = -\partial_u LF_{k,l,m}+\partial_u R_{k,l,m}$ on Q_T in the last equality. Since

$$\mathcal{L}_{\mathrm{BR}}\circ\partial_u = \partial_u\circ\mathcal{L}_{\mathrm{BR}},$$

we obtain that :

$$\begin{aligned}I_s &= \int_{t_i}^{s}\left(b(\bar{X}_q,\bar{U}_q)-b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\right)\frac{\partial}{\partial u}(\mathcal{L}_{\mathrm{BR}}-L)F_{k,l,m}(q,\bar{X}_q,\bar{U}_q)\,dq\\ &\quad+\sigma^2\int_{t_i}^{s}\partial_u b(\bar{X}_q,\bar{U}_q)\frac{\partial^2}{\partial u^2}F_{k,l,m}(q,\bar{X}_q,\bar{U}_q)\,dq\\ &\quad+\int_{t_i}^{s}\left[\left(\bar{U}_{\eta(q)}\partial_x+b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\partial_u+\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right)b(\bar{X}_q,\bar{U}_q)\right]\frac{\partial}{\partial u}F_{k,l,m}(q,\bar{X}_q,\bar{U}_q)dq\\ &\quad+\int_{t_i}^{s}\left(b(\bar{X}_q,\bar{U}_q)-b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\right)\partial_u R_{k,l,m}(q,\bar{X}_q,\bar{U}_q)dq.\end{aligned} \tag{3.29}$$

Coming back to the definition of $\mathcal{L}_{\text{BR}}$ and L we have

$$\begin{aligned}
&\partial_u(\mathcal{L}_{\text{BR}} - L)F_{k,l,m}(q, \bar{X}_q, \bar{U}_q) \\
&= \partial_u[-(\bar{U}_q - \bar{U}_{\nu(q)})\partial_x F_{k,l,m}(q, \bar{X}_q, \bar{U}_q) - \Delta b_q \partial_u F_{k,l,m}(q, \bar{X}_q, \bar{U}_q)] \\
&= -\partial_x F_{k,l,m}(q, \bar{X}_q, \bar{U}_q) - \partial_u b(\bar{X}_q, \bar{U}_q)\partial_u F_{k,l,m}(q, \bar{X}_q, \bar{U}_q) \\
&\quad - (\bar{U}_q - \bar{U}_{\eta(q)})\partial^2_{xu} F_{k,l,m}(q, \bar{X}_q, \bar{U}_q) - \Delta b_q \partial^2_{uu} F_{k,l,m}(q, \bar{X}_q, \bar{U}_q)]
\end{aligned}$$

We denote by $\Delta b_q = b(\bar{X}_q, \bar{U}_q) - b(\bar{X}_{\nu(q)}, \bar{U}_{\nu(q)})$ and the previous equality results in:

$$\begin{aligned}
I_s = &- \int_{t_i}^{s} \Delta b_q \frac{\partial}{\partial x} F_{k,l,m}(q, \bar{X}_q, \bar{U}_q) dq \\
&+ \int_{t_i}^{s} \Delta b_q \left(\bar{U}_{\eta(q)} - \bar{U}_q\right) \frac{\partial^2}{\partial x \partial u} F_{k,l,m}(q, \bar{X}_q, \bar{U}_q)\, dq \\
&+ \int_{t_i}^{s} \left(\sigma^2 \partial_u b(\bar{X}_q, \bar{U}_q) - (\Delta b_q)^2\right) \frac{\partial^2}{\partial u^2} F_{k,l,m}(q, \bar{X}_q, \bar{U}_q)\, dq \\
&+ \int_{t_i}^{s} \left[\left(\bar{U}_{\eta(q)}\partial_x + \left(b(\bar{X}_{\nu(q)}, \bar{U}_{\nu(q)}) - \Delta b_q\right)\partial_u + \frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right) b(\bar{X}_q, \bar{U}_q)\right] \frac{\partial}{\partial u} F_{k,l,m}(q, \bar{X}_q, \bar{U}_q)\, dq \\
&+ \int_{t_i}^{s} \Delta b_q \partial_u R_{k,l,m}(q, \bar{X}_q, \bar{U}_q) dq.
\end{aligned} \tag{3.30}$$

Finally, an i.b.p. is applied on $\int_{t_i}^{\theta_i} I_s ds$, by noting that $(\theta_i - s)' = -1$ in order to remove the inner integral:

$$\begin{aligned}
\epsilon^{\bar{U}}_{\text{BR}}(i) = &-\mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\theta_i - s)\Delta b_s \frac{\partial}{\partial x} F_{k,l,m}(s, \bar{X}_s, \bar{U}_s) ds \\
&+ \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\theta_i - s)\Delta b_s \left(\bar{U}_{\eta(s)} - \bar{U}_s\right) \frac{\partial^2}{\partial x \partial u} F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&+ \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\theta_i - s) \left(\sigma^2 \partial_u b(\bar{X}_s, \bar{U}_s) - (\Delta b_s)^2\right) \frac{\partial^2}{\partial u^2} F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&+ \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\theta_i - s) \left[\left(\bar{U}_{\eta(s)}\partial_x + \left(b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)}) - \Delta b_s\right)\partial_u + \frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right) b(\bar{X}_s, \bar{U}_s)\right] \\
&\qquad \times \frac{\partial}{\partial u} F_{k,l,m}(s, \bar{X}_s, \bar{U}_s) ds \\
&+ \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\theta_i - s)\Delta b_s \partial_u R_{k,l,m}(s, \bar{X}_s, \bar{U}_s) ds.
\end{aligned} \tag{3.31}$$

The term $\epsilon^{\bar{U}}_{\text{NoR}}(i)$ which corresponds to the error produced by the discretization of the drift of the velocity process in the case where no collision occurs, takes the same form as the previous formula, only requiring to replace θ_i by t_{i+1}.

Error contribution after the reflection

Similar computations to the previous paragraph are used to show that error introduced by the discretization of the drift of the velocity after the collision is:

$$
\begin{aligned}
\epsilon_{\mathrm{AR}}^{\bar U}(i) &= -\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(\theta_i-s)\Delta b_s\frac{\partial}{\partial x}F_{k,l,m}(s,\bar X_s,\bar U_s)ds\\
&+\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(t_{i+1}-s)\Delta b_s\left(\mathrm{sign}(\bar Y_s)\bar U_{\eta(s)}-\bar U_s\right)\frac{\partial^2}{\partial x\partial u}F_{k,l,m}(s,\bar X_s,\bar U_s)\,ds\\
&+\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(t_{i+1}-s)\left(\sigma^2\partial_u b(\bar X_s,\bar U_s)-(\Delta b_s)^2\right)\frac{\partial^2}{\partial u^2}F_{k,l,m}(s,\bar X_s,\bar U_s)\,ds\\
&+\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(t_{i+1}-s)\left[\left(\mathrm{sign}(\bar Y_s)\bar U_{\eta(s)}\partial_x+\left(b(\bar X_{\nu(s)},\bar U_{\nu(s)})-\Delta b_s\right)\partial_u+\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right)b(\bar X_s,\bar U_s)\right]\\
&\qquad\times\frac{\partial}{\partial u}F_{k,l,m}(s,\bar X_s,\bar U_s)ds\\
&+\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(t_{i+1}-s)\Delta b_s\partial_u R_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds.
\end{aligned}
\tag{3.32}
$$

We proceed to regroup the errors in the drift of the velocity before and after the collision by introducing the following function $\nu^R\colon\mathbb{R}^+\mapsto\mathbb{R}^+$ defined as:

$$
\nu^R(t)=\begin{cases} t_{i+1} & \text{if } \theta_i=t_i\\ \theta_i & \text{if } t\in[t_i,\theta_i) \text{ and } \theta_i\in(t_i,t_{i+1})\\ t_{i+1} & \text{if } t\in[\theta_i,t_{i+1}) \text{ and } \theta_i\in(t_i,t_{i+1})\end{cases}
\tag{3.33}
$$

and summing up (3.31) and (3.32) on all intervals for $i=0$ to $N-1$:

$$
\begin{aligned}
&\sum_{i=0}^{N-1}\left(\epsilon_{\mathrm{BR}}^{\bar U}(i)+\epsilon_{\mathrm{AR}}^{\bar U}(i)+\epsilon_{\mathrm{NoR}}^{\bar U}(i)\right)=\mathbb{E}\int_0^T\left(b(\bar X_s,\bar U_s)-b(\bar X_{\nu(s)},\bar U_{\nu(s)})\right)\partial_u f(s,\bar X_s,\bar U_s)\,ds=\\
&=-\mathbb{E}\int_0^T(\nu^R(s)-s)\Delta b_s\frac{\partial}{\partial x}F_{k,l,m}(s,\bar X_s,\bar U_s)ds\\
&\quad+\mathbb{E}\int_0^T(\nu^R(s)-s)\Delta b_s\left(\mathrm{sign}(\bar Y_s)\bar U_{\eta(s)}-\bar U_s\right)\frac{\partial^2}{\partial x\partial u}F_{k,l,m}(s,\bar X_s,\bar U_s)\,ds\\
&\quad+\mathbb{E}\int_0^T(\nu^R(s)-s)\left(\sigma^2\partial_u b(\bar X_s,\bar U_s)-(\Delta b_s)^2\right)\frac{\partial^2}{\partial u^2}F_{k,l,m}(s,\bar X_s,\bar U_s)\,ds\\
&\quad+\mathbb{E}\int_0^T(\nu^R(s)-s)\left[\left(\mathrm{sign}(\bar Y_s)\bar U_{\eta(s)}\partial_x+\left(b(\bar X_{\nu(s)},\bar U_{\nu(s)})-\Delta b_s\right)\partial_u+\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right)b(\bar X_s,\bar U_s)\right]\\
&\qquad\times\frac{\partial}{\partial u}F_{k,l,m}(s,\bar X_s,\bar U_s)\,ds\\
&\quad+\mathbb{E}\int_0^T(\nu^R(s)-s)\Delta b_s\partial_u R_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds.
\end{aligned}
\tag{3.34}
$$

3.4 Bounds on the global error

To obtain the bounds on the error, we rely on theorem 2.1. In order to obtain L^2 norms, we integrate w.r.t. to the distribution of the discretised process. A simple case where this distribution is explicit is the one without drift on the velocity component, so we apply Girsanov's theorem to remove this drift. We introduce a new probability measure $\bar{\mathbb{Q}}$ defined using Girsanov's theorem:

$$
\left.\frac{d\mathbb{P}}{d\bar{\mathbb{Q}}}\right|_{\mathcal{F}_T}=\mathcal{Z}_T=\exp\left(\int_0^T b(\bar x_s,\bar u_s)\,dW_s^0-\frac{1}{2}\int_0^T b^2(\bar x_s,\bar u_s)\,ds\right),
$$

where $(W_t^0)_{0<t<T}$ is a Brownian motion under $\bar{\mathbb{Q}}$. Since b is bounded, this means that the martingale $(\mathcal{Z}_t)_{0\leq t\leq T}$ admits moments of all orders.

We recall that for any $t\in[0,T]$, $|\nu^R(t)-t|\leq \Delta t$ and considering the first term of the equality (3.34), we have that:

$$\left|\mathbb{E}\int_0^T(\nu^R(s)-s)\Delta b_s\frac{\partial}{\partial x}F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)ds\right|\leq C_b\int_0^T\mathbb{E}\left|\frac{\partial}{\partial x}F_{k,l,m}\right|(s,\bar{X}_s,\bar{U}_s)ds\times\Delta t\,. \tag{3.35}$$

And we have that

$$\begin{aligned}\mathbb{E}\left|\frac{\partial}{\partial x}F_{k,l,m}\right|(s,\bar{X}_s,\bar{U}_s)&=\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_s\left|\frac{\partial}{\partial x}F_{k,l,m}\right|(s,\bar{x}_s,\bar{u}_s)\\&\leq\left(\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_s^2\right)^{\frac12}\left(\mathbb{E}_{\bar{\mathbb{Q}}}\left|\frac{\partial}{\partial x}F_{k,l,m}\right|^2(s,\bar{x}_s,\bar{u}_s)\right)^{\frac12}\\&\leq C_{\mu_0,b,\sigma,T}\left\|\frac{\partial}{\partial x}F_{k,l,m}\right\|_{L^2(\mathcal{D}\times\mathbb{R})}\end{aligned} \tag{3.36}$$

while for the second term of the equality (3.34) we have that

$$\begin{aligned}&\left|\mathbb{E}\int_0^T(\nu^R(s)-s)\left(b(\bar{X}_s,\bar{U}_s)-b(\bar{X}_{\nu(s)},\bar{U}_{\nu(s)})\right)\left(\mathrm{sign}(\bar{Y}_s)\bar{U}_{\eta(s)}-\bar{U}_s\right)\frac{\partial^2}{\partial x\partial u}F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\right|\\&\leq C_b\int_0^T\mathbb{E}\left(\left|\bar{U}_{\eta(s)}\right|+\left|\bar{U}_s\right|\right)\left|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right|(s,\bar{X}_s,\bar{U}_s)\,ds\times\Delta t\end{aligned} \tag{3.37}$$

where C_b depends only on the upper bound of the drift b. By choosing two positive numbers p,q such that $q>2$ and $\frac1p+\frac1q=1$, by Hölder's inequality:

$$\begin{aligned}&\mathbb{E}\left|\bar{U}_{\eta(s)}\right|\left|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right|(s,\bar{X}_s,\bar{U}_s)\leq\left(\mathbb{E}\left|\bar{U}_{\eta(s)}\right|^q\right)^{\frac1q}\left(\mathbb{E}\left|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right|^p(s,\bar{X}_s,\bar{U}_s)\right)^{\frac1p}\\&\leq\left(\mathbb{E}_{\bar{\mathbb{Q}}}\bar{u}^q_{\eta(s)}\mathcal{Z}_{\eta(s)}\right)^{\frac1q}\left(\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_s\left|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right|^p(s,\bar{X}_s,\bar{U}_s)\right)^{\frac1p}\\&\leq\left(\mathbb{E}_{\bar{\mathbb{Q}}}\bar{u}^{2q}_{\eta(s)}\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}^2_{\eta(s)}\right)^{\frac{1}{2q}}\left(\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_s^{\frac{2}{2-p}}\right)^{\frac{2-p}{2p}}\left(\mathbb{E}_{\bar{\mathbb{Q}}}\left|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right|^2(s,\bar{x}_s,\bar{u}_s)\right)^{\frac12}\\&\leq C_{\mu_0,b,\sigma,T,p,q}\left\|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}(s,\cdot,\cdot)\right\|_{L^2(\mathcal{D}\times\mathbb{R})}\end{aligned} \tag{3.38}$$

where $C_{\mu_0,b,\sigma,T,p,q}$ depends on the $2q-$moment of μ_0, the bound on b, the diffusion term σ, final time T, p and q. We perform the same calculation for the second term of (3.37) and obtain that:

$$\begin{aligned}&\left|\mathbb{E}\int_0^T(\nu^R(s)-s)\left(b(\bar{X}_s,\bar{U}_s)-b(\bar{X}_{\nu(s)},\bar{U}_{\nu(s)})\right)\left(\mathrm{sign}(\bar{Y}_s)\bar{U}_{\eta(s)}-\bar{U}_s\right)\frac{\partial^2}{\partial x\partial u}F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\right|\\&\leq C_{\mu_0,b,\sigma,T,p,q}\left\|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right\|_{L^2(Q_T)}\times\Delta t\leq C_{\mu_0,b,\sigma,T,p,q}\left\|\frac{\partial^2}{\partial x\partial u}F\right\|_{L^2(Q_T)}\times\Delta t\end{aligned} \tag{3.39}$$

since $F_{k,l,m}$ is a convolution of F where $\partial_{xu}F\in L^2(Q_T)$ by Theorem 2.1.

Since $\Delta b_s=b(\bar{X}_s,\bar{U}_s)-b(\bar{X}_{\nu(s)},\bar{U}_{\nu(s)})$, is such that $|\Delta b_s|\leq 2\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})}$ so the third term of the inequality (3.34) is bounded by:

$$\begin{aligned}&\left|\mathbb{E}\int_0^T(\nu^R(s)-s)\left(\sigma^2\frac{\partial b}{\partial u}(\bar{X}_s,\bar{U}_s)-(\Delta b_s)^2\right)\frac{\partial^2F_{k,l,m}}{\partial u^2}(s,\bar{X}_s,\bar{U}_s)\,ds\right|\leq\\&\leq C_{b,\partial_u b,\sigma,T}\int_0^T\mathbb{E}\left|\frac{\partial^2F_{k,l,m}}{\partial u^2}\right|(s,\bar{X}_s,\bar{U}_s)\,ds\times\Delta t\end{aligned}$$

$C_{b,\partial_u b,\sigma,T}$ depends only on the L^∞ norm of b and $\partial_u b$ and on σ. By Girsanov's theorem:

$$
\begin{aligned}
\mathbb{E}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|(s,\bar{X}_s,\bar{U}_s) &= \mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_T\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|(s,\bar{x}_s,\bar{u}_s) \leq \left(\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_T^2\right)^{\frac{1}{2}}\left(\mathbb{E}_{\bar{\mathbb{Q}}}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|^2(s,\bar{x}_s,\bar{u}_s)\right)^{\frac{1}{2}} \\
&\leq \exp\left(\frac{3}{4}\|b\|_{L^\infty}^2 T\right)\left(\int_{\mathcal{D}\times\mathbb{R}}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|^2(s,\xi,\zeta)\bar{p}^c(s,\xi,\zeta)\,d\xi d\zeta\right)^{\frac{1}{2}} \\
&\leq \exp\left(\frac{3}{4}\|b\|_{L^\infty}^2 T\right)\|\bar{p}^c(s;\cdot,\cdot)\|_{L^\infty(\mathcal{D}\times\mathbb{R})}^{\frac{1}{2}}\left(\int_{\mathcal{D}\times\mathbb{R}}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|^2(s,\xi,\zeta)\,d\xi d\zeta\right)^{\frac{1}{2}} \\
&\leq \exp\left(\frac{3}{4}\|b\|_{L^\infty}^2 T\right)\|2\mu_0\|_{L^\infty(\mathcal{D}\times\mathbb{R})}^{\frac{1}{2}}\left(\int_{\mathcal{D}\times\mathbb{R}}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|^2(s,\xi,\zeta)\,d\xi d\zeta\right)^{\frac{1}{2}},
\end{aligned}
\tag{3.40}
$$

where μ_0 is the p.d.f. of the initial values so it follows ($H_{\textit{Weak Error}}$), meaning that $\mu_0\in L^\infty(\mathcal{D}\times\mathbb{R})$. Integrated on $[0,T]$, we get that:

$$
\begin{aligned}
&\int_0^T \mathbb{E}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|(s,\bar{X}_s,\bar{U}_s)\,ds \\
&\leq e^{\left(\frac{3}{2}\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})}^2 T\right)}\|2\mu_0\|_{L^\infty(\mathcal{D}\times\mathbb{R})}^{\frac{1}{2}}\int_0^T\left\|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right\|_{L^2(\mathcal{D}\times\mathbb{R})}ds \\
&\quad\leq \sqrt{T}e^{\left(\frac{3}{2}\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})}^2 T\right)}\|\mu_0\|_{L^\infty(\mathcal{D}\times\mathbb{R})}^{\frac{1}{2}}\left\|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right\|_{L^2(Q_T)} \leq \sqrt{T}e^{\left(\frac{3}{2}\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})}^2 T\right)}\|\mu_0\|_{L^\infty(\mathcal{D}\times\mathbb{R})}^{\frac{1}{2}}\left\|\frac{\partial^2 F}{\partial u^2}\right\|_{L^2(Q_T)}.
\end{aligned}
$$

Concerning the third term of (3.34):

$$
\begin{aligned}
&\left|\mathbb{E}\int_0^T(\nu^R(s)-s)\left[\left(\operatorname{sign}(\bar{Y}_s)\bar{U}_{\eta(s)}\partial_x + b(\bar{X}_{\nu(s)},\bar{U}_{\nu(s)})\partial_u + \frac{\sigma^2}{2}\partial_{uu}\right)b(\bar{X}_s,\bar{U}_s)\right]\frac{\partial}{\partial u}F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\right| \\
&\quad\leq C_{b,\partial_x b,\partial_u b,\sigma,T}\int_0^T\mathbb{E}\left(1+\left|\bar{U}_{\eta(s)}\right|\right)\left|\frac{\partial F_{k,l,m}}{\partial u}\right|(s,\bar{X}_s,\bar{U}_s)\,ds\times\Delta t \\
&\quad\leq C_{\mu_0,b,\partial_x b,\partial_u b,\sigma,T,p,q}\left\|\frac{\partial F_{k,l,m}}{\partial u}\right\|_{L^2(Q_T)}\times\Delta t = C_{\mu_0,b,\partial_x b,\partial_u b,\sigma,T,p,q}\left\|\frac{\partial F}{\partial u}\right\|_{L^2(Q_T)}\times\Delta t\,,
\end{aligned}
\tag{3.41}
$$

where $C_{\mu_0,b,\partial_x b,\partial_u b,\sigma,T,p,q}$ depends on the $2q-$moment of μ_0, the bound on b and its derivatives, on σ, final time T, on p and q.

Regarding the last term of (3.34), we use the expression of the error written in Corollary 3.1:

$$
\left|\mathbb{E}\int_0^T(\nu^R(s)-s)\Delta b_s\partial_u R_{k,l,m}[F](s,\bar{X}_s,\bar{U}_s)\,ds\right| \leq C_b\mathbb{E}\int_0^T\left|\partial_u R_{k,l,m}[F](s,\bar{X}_s,\bar{U}_s)\right|\,ds\times\Delta t \tag{3.42}
$$

and

$$
\begin{aligned}
&\mathbb{E}\int_0^T\left|\partial_u R_{k,l,m}[F](s,\bar{X}_s,\bar{U}_s)\right|\,ds \leq \mathbb{E}\int_0^T\left|\partial_u R^{\mathrm{Sp}}_{k,l,m}[F](s,\bar{X}_s,\bar{U}_s)\right|\,ds + \mathbb{E}\int_0^T\left|\partial_u R^{\mathrm{Tm}}_{k,l,m}[F](s,\bar{X}_s,\bar{U}_s)\right|\,ds \\
&\leq \mathbb{E}\int_0^T\left|\left(\frac{\partial^2}{\partial u\partial x}F*((ug_m)\rho_l\beta_k)\right)\right|(s,\bar{X}_s,\bar{U}_s)\,ds + \mathbb{E}\int_0^T\left|\frac{\partial}{\partial u}b(\bar{X}_s,\bar{U}_s)\left(\frac{\partial}{\partial u}F*(g_m\rho_l\beta_k)\right)\right|(s,\bar{X}_s,\bar{U}_s)\,ds \\
&\quad+\mathbb{E}\int_0^T\left|\left(\left(\frac{\partial}{\partial u}(bF)\right)*(g_m\rho_l\beta_k)\right)\right|(s,\bar{X}_s,\bar{U}_s)\,ds + \mathbb{E}\int_0^T\beta_k(s)\left|\left(\left(\frac{\partial}{\partial u}F(0,\cdot,\cdot)\right)*(g_m\rho_l)\right)\right|(\bar{X}_s,\bar{U}_s)\,ds \\
&\leq C_{\mu_0,b,\sigma,T}\left\|\frac{\partial^2 F}{\partial u\partial x}\right\|_{L^2(Q_T)} + C_{\mu_0,b,\partial_u b,\sigma,T}\left\|\frac{\partial F}{\partial u}\right\|_{L^2(Q_T)} + C_{\mu_0,b,\sigma,T}\left\|\frac{\partial^2 F}{\partial u^2}\right\|_{L^2(Q_T)} + C_{\mu_0,b,\sigma,T}\left\|\frac{\partial F(0,\cdot,\cdot)}{\partial u}\right\|_{L^\infty(\mathcal{D}\times\mathbb{R})}.
\end{aligned}
\tag{3.43}
$$

Combining all these terms gives us that:

$$\left|\sum_{i=0}^{N-1}\left(\epsilon_{\text{BR}}^{\bar{U}}(i)+\epsilon_{\text{AR}}^{\bar{U}}(i)+\epsilon_{\text{NoR}}^{\bar{U}}(i)\right)\right|$$
$$\leq C_{\mu_0,b,\partial_x b,\partial_u b,\sigma,T}\times\Delta t$$
$$\times\left(\left\|\frac{\partial F}{\partial x}\right\|_{L^2(Q_T)}+\left\|\frac{\partial F}{\partial u}\right\|_{L^2(Q_T)}+\left\|\frac{\partial^2 F}{\partial u^2}\right\|_{L^2(\mathcal{D}\times\mathbb{R})}+\left\|\frac{\partial^2 F}{\partial u\partial x}\right\|_{L^2(Q_T)}+\left\|\frac{\partial F(0,\cdot,\cdot)}{\partial u}\right\|_{L^\infty(\mathcal{D}\times\mathbb{R})}\right) \tag{3.44}$$

By the same technique we can show that (3.28) is bounded by

$$\begin{aligned}
&\sum_{i=0}^{N-1}\left(|\epsilon_{\text{BR}}^{\bar{X}}(i)|+|\epsilon_{\text{AR}}^{\bar{X}}(i)|+|\epsilon_{\text{NoR}}^{\bar{X}}(i)|\right)\\
&\leq C_{b,\partial_u b,\sigma,T}\left(\mathbb{E}\int_0^T\left(\left|\frac{\partial F_{k,l,m}}{\partial x}\right|(s,\bar{X}_s,\bar{U}_s)+\left|\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}\right|(s,\bar{X}_s,\bar{U}_s)\right)ds\right)\times\Delta t \qquad (3.45)\\
&\leq C_{\mu_0,b,\partial_u b,\sigma,T}\left(\left\|\frac{\partial F}{\partial x}\right\|_{L^2(Q_T)}+\left\|\frac{\partial^2 F}{\partial u\partial x}\right\|_{L^2(Q_T)}\right)\times\Delta t.
\end{aligned}$$

Going back to equality (3.17) and putting together all the various results:

$$\begin{aligned}
&|\mathbb{E}\psi(X_T^{t,x,u},U_T^{t,x,u})-\mathbb{E}\psi(\bar{X}_T^{t,x,u},\bar{U}_T^{t,x,u})|=\left|\sum_{i=0}^{N-1}(\epsilon_{\text{BR}}(i)+\epsilon_{\text{AR}}(i)+\epsilon_{\text{NoR}}(i))+\mathbb{E}F(0,X_0,U_0)+\epsilon_{k,l,m}^{\text{Reg}}\right|\\
&=\left|\sum_{i=0}^{N-1}\left(\epsilon_{\text{BR}}^{\bar{X}}(i)+\epsilon_{\text{BR}}^{\bar{U}}(i)+\epsilon_{\text{AR}}^{\bar{X}}(i)+\epsilon_{\text{AR}}^{\bar{U}}(i)+\epsilon_{\text{NoR}}^{\bar{X}}(i)+\epsilon_{\text{NoR}}^{\bar{U}}(i)\right)+\epsilon_{k,l,m}^{\text{Reg}}\right.\\
&\qquad\left.-\,\mathbb{E}\int_0^T R_{k,l,m}^{\text{Sp}}[F](s,\bar{X}_s,\bar{U}_s)\,ds+\mathbb{E}F(0,X_0,U_0)-\mathbb{E}\int_0^{\Delta t}R_{k,l,m}^{\text{Tm}}[F](s,\bar{X}_s,\bar{U}_s)\,ds\right|\\
&\leq\sum_{i=0}^{N-1}\left(|\epsilon_{\text{BR}}^{\bar{X}}(i)|+|\epsilon_{\text{AR}}^{\bar{X}}(i)|+|\epsilon_{\text{NoR}}^{\bar{X}}(i)|\right)+\left|\sum_{i=0}^{N-1}\left(\epsilon_{\text{BR}}^{\bar{U}}(i)+\epsilon_{\text{AR}}^{\bar{U}}(i)+\epsilon_{\text{NoR}}^{\bar{U}}(i)\right)\right|+\left|\epsilon_{k,l,m}^{\text{Reg}}\right|\\
&\quad+\left|\mathbb{E}\int_0^T R_{k,l,m}^{\text{Sp}}[F](s,\bar{X}_s,\bar{U}_s)\,ds\right|+\left|\mathbb{E}F(0,X_0,U_0)-\mathbb{E}\int_0^{\Delta t}R_{k,l,m}^{\text{Tm}}[F](s,\bar{X}_s,\bar{U}_s)\,ds\right|\\
&\quad\leq C_{\sigma,b,\partial_x b,\partial_u b,\mu_0,T}\left(\left\|\frac{\partial F}{\partial u}\right\|_{L^2(Q_T)}+\left\|\frac{\partial F}{\partial x}\right\|_{L^2(Q_T)}+\left\|\frac{\partial^2 F}{\partial u^2}\right\|_{L^2(Q_T)}\right.\\
&\qquad\qquad\left.+\left\|\frac{\partial^2 F}{\partial u\partial x}\right\|_{L^2(Q_T)}+\left\|\frac{\partial F(0,\cdot,\cdot)}{\partial u}\right\|_{L^2(\mathcal{D}\times\mathbb{R})}\right)\times\Delta t\\
&\quad+\left|\epsilon_{k,l,m}^{\text{Reg}}\right|+\left|\mathbb{E}\int_0^T R_{k,l,m}^{\text{Sp}}[F](s,\bar{X}_s,\bar{U}_s)\,ds\right|+\left|\mathbb{E}F(0,X_0,U_0)-\mathbb{E}\int_0^{\Delta t}R_{k,l,m}^{\text{Tm}}[F](s,\bar{X}_s,\bar{U}_s)\,ds\right|.
\end{aligned} \tag{3.46}$$

By Lemma 3.3 term $\left|\epsilon_{k,l,m}^{\text{Reg}}\right|+\left|\mathbb{E}F(0,X_0,U_0)-\mathbb{E}\int_0^{\Delta t}R_{k,l,m}^{\text{Tm}}[F](s,\bar{X}_s,\bar{U}_s)\,ds\right|$ goes to zero as (k,l,m) go to infinity. By Lemma 5.3, the term $R_{k,l,m}^{\text{Sp}}[F]$ converges uniformly towards 0 as (k,l,m) go to infinity, if $l=m$, thus the term $\left|\mathbb{E}\int_0^T R_{k,l,m}^{\text{Sp}}[F](s,\bar{X}_s,\bar{U}_s)\,ds\right|$ also converges to 0.

We can therefore conclude that by taking $l = m$ and $(k, l, m) \to \infty$ in the inequality (3.46) we obtain that

$$
\begin{aligned}
&\left|\mathbb{E}\psi(X_T^{t,x,u}, U_T^{t,x,u}) - \mathbb{E}\psi(\bar{X}_T^{t,x,u}, \bar{U}_T^{t,x,u})\right| \\
&\leq \Delta t \times C_{\sigma,b,\partial_x b,\partial_u b,\mu_0,T} \\
&\quad \times \left(\left\| \frac{\partial F}{\partial u} \right\|_{L^2(Q_T)} + \left\| \frac{\partial F}{\partial x} \right\|_{L^2(Q_T)} + \left\| \frac{\partial^2 F}{\partial u^2} \right\|_{L^2(Q_T)} + \left\| \frac{\partial^2 F}{\partial u \partial x} \right\|_{L^2(Q_T)} + \left\| \frac{\partial F(0,\cdot,\cdot)}{\partial u} \right\|_{L^\infty(\mathcal{D}\times\mathbb{R})} \right).
\end{aligned} \tag{3.47}
$$

This ends the proof of Theorem 1.6: the weak error of our scheme converges at least linearly in the time discretization step Δt.

4 Regularity of the flow of the free Langevin process

In this section we prove the regularity result up to the first order of the F function stated in Theorem 2.1. The results are stated in Lemma 4.3 and Lemma 4.6.

They are based on the study of the regularity of the flow in sens of Bouleau and Hirsh, for the free Lagrangian process first, for it confined version then.

We consider the free Langevin process $(Y_t, V_t) \in \mathbb{R}^d \times \mathbb{R}^d$ which verifies the equation:

$$
\begin{cases}
Y_t = x + \displaystyle\int_0^t V_s^{x,u}\, ds, \\
V_t = u + \sigma \widetilde{W}_t + \displaystyle\int_0^t \widetilde{b}(Y_s^{x,u}, V_s^{x,u})\, ds,
\end{cases} \tag{4.1}
$$

where $(x, u) \in \mathcal{D} \times \mathbb{R}^d$ and $\widetilde{b}\colon \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ defined as:

$$
\widetilde{b}(x,u) := (b', \operatorname{sign}(x^{(d)}) b^{(d)}) \left((x', |x^{(d)}|), (u', \operatorname{sign}(x^{(d)}) u^{(d)}) \right). \tag{4.2}
$$

We recall the following notation: for any $x \in \mathbb{R}^d$, we write $x = (x', x^{(d)})$ where x' are the first $(d-1)$ coordinates of x and $x^{(d)}$ the d$^{\text{th}}$ component.

The result in [8] shows that the process $(Y_t^{x,u}, V_t^{x,u})_{t\geq 0}$ admits a derivative in the sense of distributions w.r.t. the initial conditions (x, u). This result allows us to state that the gradients $\nabla_x F$ and $\nabla_u F$ in Theorem 2.1 are well defined . We reproduce their technique and arguments in this section. It involves an augmentation of the probability space to include the initial conditions and a modified SDE on the new probability space. The modified SDE respects a weaker uniqueness condition which allows to perform some operations that are not allowed on the original SDE (4.1).

4.1 Derivability of the flow in the sens of Bouleau and Hirsch

We recall the notations and results of Bouleau and Hirsch in [8] for a general process $(X_t)_{0\leq t\leq T}$ that is a solution of the stochastic differential equation:

$$
X_t^x = x + \int_0^t b(X_s^x)\, ds + \int_0^t \sigma(X_s^x)\, dW_s \tag{4.3}
$$

where the functions b and σ are Lipschitz with, at most, linear increase. Let $\Omega = \mathcal{C}_0(\mathbb{R}_+, \mathbb{R}^d)$, the Wiener space of continuous functions ω such that $\omega(0) = 0$ equipped with the metric of the uniform convergence on compacts. $\mathcal{F}$ is the Borel $\sigma-$algebra over Ω and $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. The canonical process is defined as $W_t(\omega) = \omega(t)$ for all $t \geq 0$. Then $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W_t)$ is a Brownian motion. The authors enlarge the probability space as $\widetilde{\Omega} = \mathbb{R}^d \times \Omega$ and $\widetilde{\mathcal{F}}$ the Borel $\sigma-$algebra over $\widetilde{\Omega}$. $\widetilde{\mathbb{P}}$ is the product measure $hdx \otimes \mathbb{P}$ where h is a probability density that has a second order moment. The canonical process is therefore $\widetilde{W}_t(x,\omega) = W_t$ with natural filtration $\widetilde{\mathcal{F}}_t$ which is augmented by the $\widetilde{P}-$negligible sets of $\widetilde{\mathcal{F}}$. Then $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}}, \widetilde{W}_t)$ is the canonical Brownian motion starting from 0. Let $e_1, \ldots, e_n$ be the canonical basis in $\mathbb{R}^d$. For every i in $\{1, \ldots, d\}$, the Dirichlet space $\widetilde{D}_i$ is defined as:

$$\widetilde{D}_i = \left\{ \begin{array}{l} u\colon \widetilde{\Omega} \mapsto \mathbb{R},\ \exists \widetilde{u}\colon \widetilde{\Omega} \mapsto \mathbb{R} \text{ Borel measurable s.t. } u = \widetilde{u}, \widetilde{\mathbb{P}} - \text{a.e. and} \\ \forall (x,\omega) \in \widetilde{\Omega}, t \mapsto \widetilde{u}(x + te_i, \omega) \text{ is locally absolutely continuous} \end{array} \right\}$$

so $\widetilde{D}_i$ can be considered as a set of classes w.r.t. $\widetilde{\mathbb{P}}-$a.e. equality. If u is in $\widetilde{D}_i$ and $\widetilde{u}$ is associated with it according to the above definition, then:

$$\nabla_i u(x,\omega) = \lim_{t\to 0} \frac{\widetilde{u}(x+te_i,\omega) - \widetilde{u}(x,\omega)}{t}.$$

Let $\widetilde{D}$ be the Dirichlet space defined as:

$$\widetilde{D} = \left\{ u \in L^2(\widetilde{\mathbb{P}}) \bigcap \left(\bigcap_{i=1}^{d} \widetilde{D}_i \right); \forall 1 \le i \le d, \nabla_i u \in L^2(\widetilde{\mathbb{P}}) \right\}$$

equipped with the norm:

$$\|u\|_{\widetilde{D}} = \left(\int_{\widetilde{\Omega}} u^2 \, d\widetilde{\mathbb{P}} + \sum_{i=1}^{d} \int_{\widetilde{\Omega}} (\nabla_i u)^2 \, d\widetilde{\mathbb{P}} \right)^{\frac{1}{2}}.$$

We also consider the space $D = \{f \in L^2(hdx); \forall 1 \le j \le d \frac{\partial}{\partial x_j} f \in L^2(hdx)\}$ equipped with its usual norm. We introduce the process $(\widetilde{X}_t^x)_{0\le t\le T}$ defined on the space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\ge 0}, \widetilde{\mathbb{P}}, \widetilde{W}_t)$ that solves the stochastic differential equation:

$$\widetilde{X}_t^x = x + \int_0^t b(\widetilde{X}_s^x)\, ds + \int_0^t \sigma(\widetilde{X}_s^x)\, d\widetilde{W}_s. \tag{4.4}$$

It can be shown that for every $0 \le t \le T$, $\widetilde{X}_t = X_t^x$, $\widetilde{\mathbb{P}}-$almost surely.

Theorem 4.1 ([8])**.**
(i) For $\mathbb{P}-$almost every ω, for all $0 \le t \le T$, $X_t^{\cdot}(\omega) \in D^d \subset (H^1_{loc}(\mathbb{R}^d))^d$
(ii) There exists a $(\widetilde{\mathcal{F}}_t)-$adapted $GL_d(\mathbb{R})-$valued continuous process $(M_t)_{0\le t\le T}$ such that, for $\widetilde{\mathbb{P}}-$almost every ω,

$$\forall t \le T \frac{\partial}{\partial x}\left(X_t^x(\omega)\right) = M_t(x,\omega) \quad dx - a.e.$$

where $\frac{\partial}{\partial x}$ denotes the derivative in the distribution sense.

And also:

Lemma 4.2. *$(M_t)_{0\le t\le T}$ is the $\mathbb{R}^{d\times d}-$values $(\widetilde{\mathcal{F}}_t)-$ adapted continuous solution of the linear sde:*

$$M_t^i = e_i + \int_0^t b_x(\widetilde{X}_s^x) M_s^i \, ds + \sum_{j=1}^{d} \int_0^t \sigma_x^j(\widetilde{X}_s^x) M_s^i \, d\widetilde{W}_s^j$$

for all $1 \le i \le d$, where b_x and σ_x^j are versions of the almost everywhere derivatives of b and σ^j.

4.2 Application to the free Langevin process

We apply theorem 4.1 and lemma 4.2 to the process (4.1), since the function $\widetilde{b}$ is Lipschitz with linear growth and σ is a constant. Then there exists $(\widetilde{\mathcal{F}}_t)-$adapted processes, parametrised by $(x,u) \in \mathbb{R}^d \times \mathbb{R}^d$, $(M_t^Y(x,u))$, $(M_t^V(x,u))$, $(N_t^Y(x,u))$, $(N_t^V(x,u))$ such that:

$$\begin{cases} \nabla_x Y_t^{x,u} = M_t^Y(x,u) \\ \nabla_x V_t^{x,u} = M_t^V(x,u) \\ \nabla_u Y_t^{x,u} = N_t^Y(x,u) \\ \nabla_u V_t^{x,u} = N_t^V(x,u) \end{cases} \tag{4.5}$$

where

$$\begin{cases} M_t^Y(x,u) = I_d + \int_0^t M_s^V(x,u)\,ds \\ M_t^V(x,u) = \int_0^t \widetilde{b}_x(\widetilde{Y}_s^{x,u},\widetilde{V}_s^{x,u})M_s^Y(x,u)\,ds + \int_0^t \widetilde{b}_u(\widetilde{Y}_s^{x,u},\widetilde{V}_s^{x,u})M_s^V(x,u)\,ds \\ N_t^Y(x,u) = \int_0^t N_s^V(x,u)\,ds \\ N_t^V(x,u) = I_d + \int_0^t \widetilde{b}_x(\widetilde{Y}_s^{x,u},\widetilde{V}_s^{x,u})N_s^Y(x,u)\,ds + \int_0^t \widetilde{b}_u(\widetilde{Y}_s^{x,u},\widetilde{V}_s^{x,u})N_s^V(x,u)\,ds \end{cases} \tag{4.6}$$

where $\widetilde{b}_x$ and $\widetilde{b}_u$ are versions of the almost everywhere derivatives in x and u of $\widetilde{b}$. I_d is the identity in dimension d. Since

$$\widetilde{b}(x,u) = (b', \mathrm{sign}(x^{(d)})b^{(d)})\left((x',|x^{(d)}|),(u',\mathrm{sign}(x^{(d)})u^{(d)})\right)$$

we take $\widetilde{b}_x(x,u) = (\nabla_{x'}\widetilde{b}, \partial_{x^{(d)}}\widetilde{b})(x,u)$ and $\widetilde{b}_u(x,u) = (\nabla_{u'}\widetilde{b}, \partial_{u^{(d)}}\widetilde{b})(x,u)$, where

$$\begin{cases} \nabla_{x'}\widetilde{b}(x,u) = (\nabla_{x'}b', \mathrm{sign}(x^{(d)})\nabla_{x'}b^{(d)})\left((x',|x^{(d)}|),(u',\mathrm{sign}(x^{(d)})u^{(d)})\right) \\ \partial_{x^{(d)}}\widetilde{b}(x,u) = (\mathrm{sign}(x^{(d)})\partial_{x^{(d)}}b', \partial_{x^{(d)}}b^{(d)})\left((x',|x^{(d)}|),(u',\mathrm{sign}(x^{(d)}u^{(d)}))\right) \\ \nabla_{u'}\widetilde{b}(x,u) = (\nabla_{u'}b', \mathrm{sign}(x^{(d)})\nabla_{u'}b^{(d)})\left((x',|x^{(d)}|),(u',\mathrm{sign}(x^{(d)})u^{(d)})\right) \\ \partial_{u^{(d)}}\widetilde{b}(x,u) = (\mathrm{sign}(x^{(d)})\partial_{u^{(d)}}b', \partial_{u^{(d)}}b^{(d)})\left((x',|x^{(d)}|),(u',\mathrm{sign}(x^{(d)})u^{(d)})\right). \end{cases} \tag{4.7}$$

4.2.1 Properties of the weak derivatives in a no-drift setting

Let $(z_t^{x,u},\eta_t^{x,u})_{0\le t\le T}$ be the process that solves the following SDE under a new probability measure $\mathbb{P}_{z,\eta}$:

$$\begin{cases} z_t^{x,u} = x + \int_0^t \eta_s^u\,ds \\ \eta_t^u = u + \sigma\check{W}_t \end{cases} \tag{4.8}$$

where $(\check{W}_t)_{0\le t\le T}$ is a Brownian motion under the new probability. We also consider the following processes defined by the equations:

$$\begin{cases} \check{M}_t^Y(x,u) = I_d + \int_0^t \check{M}_s^V(x,u)\,ds \\ \check{M}_t^V(x,u) = \int_0^t \widetilde{b}_x(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{M}_s^Y(x,u)\,ds + \int_0^t \widetilde{b}_u(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{M}_s^V(x,u)\,ds \\ \check{N}_t^Y(x,u) = \int_0^t \check{N}_s^V(x,u)\,ds \\ \check{N}_t^V(x,u) = I_d + \int_0^t \widetilde{b}_x(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{N}_s^Y(x,u)\,ds + \int_0^t \widetilde{b}_u(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{N}_s^V(x,u)\,ds. \end{cases} \tag{4.9}$$

We analyse the continuity at the boundary $\partial\mathcal{D}$ of the solutions of (4.9) starting with the term $\check{M}_t^V$.

Lemma 4.3. *For any $(t,u)\in[0,T]\times\mathbb{R}^d$ and $p\in[1,\infty)$, the processes $\check{M}_t^Y(\cdot,u)$, $\check{M}_t^V(\cdot,u)$, $\check{N}_t^Y(\cdot,u)$ and $\check{N}_t^V(\cdot,u)$ are continuous up to the boundary $\partial\mathcal{D}$ in norm L^p.*

Proof. This result is proved using Gronwall's lemma. The regularity of the derivatives $\widetilde{b}_x$ and $\widetilde{b}_u$ is used. The regularity of the density of the drift-less free Langevin model is used to smooth out the changes of sign when the boundary is hit.

Let $(t,x,u) \in Q_T$ and $\bar{x} \in \partial\mathcal{D}$ the projection of x on $\partial\mathcal{D}$. By the system (4.9):

$$\begin{aligned}
\left|\check{M}_t^V(x,u) - \check{M}_t^V(\bar{x},u)\right| &= \bigg|\int_0^t \widetilde{b}_x(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{M}_s^Y(x,u)\,ds + \int_0^t \widetilde{b}_u(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{M}_s^V(x,u)\,ds \\
&\qquad - \int_0^t \widetilde{b}_x(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\check{M}_s^Y(\bar{x},u)\,ds - \int_0^t \widetilde{b}_u(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\check{M}_s^V(\bar{x},u)\,ds\bigg| \\
&\le \left|\int_0^t \widetilde{b}_x(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{M}_s^Y(x,u)\,ds - \int_0^t \widetilde{b}_x(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\check{M}_s^Y(\bar{x},u)\,ds\right| + \\
&\qquad + \left|\int_0^t \widetilde{b}_u(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{M}_s^V(x,u)\,ds - \int_0^t \widetilde{b}_u(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\check{M}_s^V(\bar{x},u)\,ds\right|.
\end{aligned} \tag{4.10}$$

The first term in this sum corresponds to the first derivative in x while the second term of the sum corresponds to the first derivative in u. By (H_{PDE})-(i) and Gronwall's lemma it is easy to notice that there exists a constant $C_{\nabla_x b,\nabla_u b,T}$ such that:

$$\sup_{(t,x,u)\in\overline{Q_T}} \left(\left\|\check{M}_t^Y(x,u)\right\| + \left\|\check{M}_t^V(x,u)\right\| + \left\|\check{N}_t^Y(x,u)\right\| + \left\|\check{N}_t^V(x,u)\right\|\right) < C_{\nabla_x b,\nabla_u b,T}$$

so:

$$\begin{aligned}
&\left|\int_0^t \widetilde{b}_x(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{M}_s^Y(x,u)\,ds - \int_0^t \widetilde{b}_x(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\check{M}_s^Y(\bar{x},u)\,ds\right| \\
&\le \left|\int_0^t \widetilde{b}_x(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\left(\check{M}_s^Y(x,u) - \check{M}_s^Y(\bar{x},u)\right)\,ds\right| \\
&\qquad + \left|\int_0^t \left(\widetilde{b}_x(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u) - \widetilde{b}_x(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\right)\check{M}_s^Y(\bar{x},u)\,ds\right| \\
&\le \left\|\widetilde{b}_x\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\int_0^t \left|\check{M}_s^Y(x,u) - \check{M}_s^Y(\bar{x},u)\right|\,ds \\
&\qquad + \sup_{t\in[0,T]}\left\|\check{M}_t^Y(\bar{x},u)\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\int_0^t \left|\widetilde{b}_x(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u) - \widetilde{b}_x(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\right|\,ds \\
&\le \left\|\widetilde{b}_x\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\int_0^t\int_0^s \left|\check{M}_\theta^V(x,u) - \check{M}_\theta^V(\bar{x},u)\right|\,d\theta ds \\
&\qquad + \sup_{t\in[0,T]}\left\|\check{M}_t^Y(\bar{x},u)\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\int_0^t \left|\nabla_{x'}\widetilde{b}(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u) - \nabla_{x'}\widetilde{b}(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\right|\,ds \\
&\qquad + \sup_{t\in[0,T]}\left\|\check{M}_t^Y(\bar{x},u)\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\int_0^t \left|\partial_{x^{(d)}}\widetilde{b}(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u) - \partial_{x^{(d)}}\widetilde{b}(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\right|\,ds.
\end{aligned} \tag{4.11}$$

The second term of this inequality represents the derivatives of the drift with respect to the first $d-1$ coordinates while the third term corresponds to the derivative w.r.t the d^{th} coordinate. These two terms are analyzed separately in the following paragraphs: **The derivative on the first *d-1* directions** and **The derivative on the d^{th} direction**.

The derivative on the first *d-1* directions

Going back to the choices for the derivatives of $\widetilde{b}$ in (4.7), we have that:

$$\begin{aligned}
&\int_0^t \left|\nabla_{x'}\widetilde{b}(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u) - \nabla_{x'}\widetilde{b}(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\right|\,ds \\
&\le \int_0^t \bigg|\nabla_{x'}b'\left(\left((\widetilde{z}_s^{x,u})', \left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right) \\
&\qquad - \nabla_{x'}b'\left(\left((\widetilde{z}_s^{\bar{x},u})', \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\bigg|\,ds \\
&\quad + \int_0^t \bigg|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)\nabla_{x'}b^{(d)}\left(\left((\widetilde{z}_s^{x,u})', \left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right) \\
&\qquad - \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\nabla_{x'}b^{(d)}\left(\left((\widetilde{z}_s^{\bar{x},u})', \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\bigg|\,ds.
\end{aligned} \tag{4.12}$$

We recall that the derivative w.r.t. x of the drift b is Lipschitz continuous which we denote as $L_{\nabla_x b}$ its Lipschitz constant. Also the $d-$dimensional free Langevin process with no drift defined in (4.8) can be considered as being d independent $1-$dimensional free Langevin processes. This results in:

$$\begin{aligned}
&\int_0^t \Big|\nabla_{x'} b'\left(\left((\widetilde{z}_s^{x,u})', \left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right) \\
&\quad - \nabla_{x'} b'\left(\left((\widetilde{z}_s^{\bar{x},u})', \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\Big| \, ds \\
&\leq L_{\nabla_x b}\left(\int_0^t \left|\left|(\widetilde{z}_s^{x,u})^{(d)}\right| - \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right| ds + \int_0^t \left|(\widetilde{\eta}_s^u)^{(d)}\right| \left|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds\right) \\
&\leq L_{\nabla_x b}\left(\int_0^t \left|(\widetilde{z}_s^{x,u})^{(d)} - (\widetilde{z}_s^{\bar{x},u})^{(d)}\right| ds + \int_0^t \left|(\widetilde{\eta}_s^u)^{(d)}\right| \left|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds\right).
\end{aligned} \tag{4.13}$$

For the second integral of (4.12), we have by the boundedness of $\nabla_x b$:

$$\begin{aligned}
&\int_0^t \Big|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)\nabla_{x'} b^{(d)}\left(\left((\widetilde{z}_s^{x,u})', \left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right) \\
&\qquad - \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\nabla_{x'} b^{(d)}\left(\left((\widetilde{z}_s^{\bar{x},u})', \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\Big| \, ds \\
&\leq \|\nabla_x b\|_{L^\infty(Q_T,\mathbb{R}^{2d})}\int_0^t \left|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds \\
&\quad + \int_0^t \Big|\nabla_{x'} b^{(d)}\left(\left((\widetilde{z}_s^{x,u})', \left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right) \\
&\qquad - \nabla_{x'} b^{(d)}\left(\left((\widetilde{z}_s^{\bar{x},u})', \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\Big| \, ds \\
&\leq C_{\nabla_x b}\int_0^t \left(1 + \left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds + L_{\nabla_x b}\int_0^t \left|(\widetilde{z}_s^{x,u})^{(d)} - (\widetilde{z}_s^{\bar{x},u})^{(d)}\right| ds
\end{aligned} \tag{4.14}$$

where $C_{\nabla_x b} = \max\{L_{\nabla_x b}, \|\nabla_x b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\}$. Combining these two previous inequalities and using the definition of the free Langevin model with no drift (4.8), we go back to inequality (4.12) to obtain:

$$\begin{aligned}
&\int_0^t \left|\nabla_{x'}\widetilde{b}(\widetilde{z}_s^{x,u}, \widetilde{\eta}_s^u) - \nabla_{x'}\widetilde{b}(\widetilde{z}_s^{\bar{x},u}, \widetilde{\eta}_s^u)\right| ds \\
&\leq C_{\nabla_x b}\left(\int_0^t \left(1 + \left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds + t\,|x - \bar{x}|\right)
\end{aligned} \tag{4.15}$$

where $C_{\nabla_x b} = 2\max\{L_{\nabla_x b}, \|\nabla_x b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^{2d})}\}$.

The derivative on the d^{th} direction

We develop the third term of the inequality (4.11) based on the same arguments used in the previous section:

$$\begin{aligned}
&\int_0^t \left|\partial_{x^{(d)}}\widetilde{b}(\widetilde{z}_s^{x,u}, \widetilde{\eta}_s^u) - \partial_{x^{(d)}}\widetilde{b}(\widetilde{z}_s^{\bar{x},u}, \widetilde{\eta}_s^u)\right| ds \\
&\leq \int_0^t \Big|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)\partial_{x^{(d)}} b'\left(\left((\widetilde{z}_s^{x,u})', \left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right) \\
&\qquad - \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\partial_{x^{(d)}} b'\left(\left((\widetilde{z}_s^{\bar{x},u})', \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\Big| \, ds \\
&\quad + \int_0^t \Big|\partial_{x^{(d)}} b^{(d)}\left(\left((\widetilde{z}_s^{x,u})', \left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right) \\
&\qquad - \partial_{x^{(d)}} b^{(d)}\left(\left((\widetilde{z}_s^{\bar{x},u})', \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\Big| \, ds \\
&\leq C_{\nabla_x b}\left(\int_0^t \left(1 + \left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds + t\,|x - \bar{x}|\right)
\end{aligned} \tag{4.16}$$

where $C_{\nabla_x b} = 2\max\{L_{\nabla_x b}, \|\nabla_x b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^{2d})}\}$. Finally, from inequality (4.11):

$$\begin{aligned}
&\left|\int_0^t \widetilde{b}_x(\widetilde{z}_s^{x,u}, \widetilde{\eta}_s^u)\check{M}_s^Y(x,u)\,ds - \int_0^t \widetilde{b}_x(\widetilde{z}_s^{\bar{x},u}, \widetilde{\eta}_s^u)\check{M}_s^Y(\bar{x},u)\,ds\right| \\
&\leq C_{\nabla_x b}\left(T\int_0^t \left|\check{M}_\theta^V(x,u) - \check{M}_\theta^V(\bar{x},u)\right|\,d\theta + \int_0^t \left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right|\,ds\right) \\
&\quad + C_{\nabla_x b} t\,|x-\bar{x}|
\end{aligned} \tag{4.17}$$

where $C_{\nabla_x b} = 2\max\{L_{\nabla_x b}, \|\nabla_x b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^{2d})}\}$.

Similar calculations show that the second term from (4.10) is bounded by:

$$\begin{aligned}
&\left|\int_0^t \widetilde{b}_u(\widetilde{z}_s^{x,u}, \widetilde{\eta}_s^u)\check{M}_s^V(x,u)\,ds - \int_0^t \widetilde{b}_u(\widetilde{z}_s^{\bar{x},u}, \widetilde{\eta}_s^u)\check{M}_s^V(\bar{x},u)\,ds\right| \\
&\leq \|\nabla_u b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d}}\int_0^t \left|\check{M}_s^V(x,u) - \check{M}_s^V(\bar{x},u)\right|\,ds + C_{\nabla_u b} t\,|x-\bar{x}| \\
&\quad + C_{\nabla_u b}\int_0^t \left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right|\,ds
\end{aligned} \tag{4.18}$$

where $C_{\nabla_u b} = 2\max\{L_{\nabla_u b}, \|\nabla_u b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\}$. Combining these inequalities gives for (4.10):

$$\begin{aligned}
\left|\check{M}_t^V(x,u) - \check{M}_t^V(\bar{x},u)\right| &\leq C_{\nabla_x b,\nabla_u b,T}\left(\int_0^t \left|\check{M}_s^V(x,u) - \check{M}_s^V(\bar{x},u)\right|\,ds + |x-\bar{x}|\right) \\
&+ C_{\nabla_x b,\nabla_u b,T}\int_0^t \left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right|\,ds\,.
\end{aligned} \tag{4.19}$$

Taking the expectation under $\mathbb{P}_{z,\eta}$ of the previous equation, we obtain for any $p \geq 1$:

$$\begin{aligned}
\mathbb{E}_{z,\eta}\left|\check{M}_t^V(x,u) - \check{M}_t^V(\bar{x},u)\right|^p &\leq C^p_{\nabla_x b,\nabla_u b,T}3^{p-1}\left(\mathbb{E}_{z,\eta}\int_0^t \left|\check{M}_s^V(x,u) - \check{M}_s^V(\bar{x},u)\right|^p\,ds + |x-\bar{x}|^p\right. \\
&\left. + \mathbb{E}_{z,\eta}\int_0^t \left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)^p\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right|^p\,ds\right).
\end{aligned} \tag{4.20}$$

Gronwall's lemma gives that:

$$\begin{aligned}
\mathbb{E}_{z,\eta}\left|\check{M}_t^V(x,u) - \check{M}_t^V(\bar{x},u)\right|^p &\leq C_{\nabla_x b,\nabla_u b,T,p}e^{C_{\nabla_x b,\nabla_u b,T,p}}\Big(|x-\bar{x}| \\
&+ \int_0^t \mathbb{E}_{z,\eta}\left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)^p\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right|^p\,ds\Big)
\end{aligned} \tag{4.21}$$

where $C_{\nabla_x b,\nabla_u b,T,p}$ depends on $\nabla_x b$, $\nabla_u b$, T and p. Recalling that components of the d dimensional drift-less free Langevin model in (4.8) are independent:

$$
\begin{aligned}
&\mathbb{E}_{z,\eta}\left(1+\left|(\widetilde{\eta}^u_s)^{(d)}\right|\right)^p\left|\text{sign}\left((\widetilde{z}^{x,u}_s)^{(d)}\right)-\text{sign}\left((\widetilde{z}^{\bar{x},u}_s)^{(d)}\right)\right|^p\\
&\le\left(\mathbb{E}_{z,\eta}\left(1+\left|u^{(d)}+\sigma\check{W}^{(d)}_t\right|\right)^{2p}\right)^{\frac{1}{2}}\left(\mathbb{E}_{z,\eta}\left|\text{sign}\left((\widetilde{z}^{x,u}_s)^{(d)}\right)-\text{sign}\left((\widetilde{z}^{\bar{x},u}_s)^{(d)}\right)\right|^{2p}\right)^{\frac{1}{2}}\\
&\le 3^{p-\frac{1}{2}}\left(1+|u^{(d)}|^{2p}+t^p\sigma^{2p}\frac{2p!}{2^p p!}\right)^{\frac{1}{2}}\\
&\quad\times\left(\mathbb{E}_{z,\eta}\left|\text{sign}\left(x^{(d)}+u^{(d)}s+\sigma\int_0^s\check{W}^{(d)}_\theta\,d\theta\right)-\text{sign}\left(u^{(d)}s+\sigma\int_0^s\check{W}^{(d)}_\theta\,d\theta\right)\right|^{2p}\right)^{\frac{1}{2}}\\
&\le C_{u,\sigma,T,p}\left(2^{2p}\mathbb{P}_{z,\eta}\left(u^{(d)}s+\sigma\int_0^s\check{W}^{(d)}_\theta\,d\theta\le 0\le x^{(d)}+u^{(d)}s+\sigma\int_0^s\check{W}^{(d)}_\theta\,d\theta\right)\right)^{\frac{1}{2}}\\
&\le C_{u,\sigma,T,p}\left(\text{erf}\left(\sqrt{\frac{3}{2}}\frac{x^{(d)}+su^{(d)}}{\sigma s\sqrt{s}}\right)-\text{erf}\left(\sqrt{\frac{3}{2}}\frac{u}{\sigma\sqrt{s}}\right)\right)^{\frac{1}{2}}
\end{aligned}
\tag{4.22}
$$

where $C_{u,\sigma,T,p}$ depends on u, σ, T and p, and erf is the error function.

Lebesgue dominated convergence theorem gives that bound in (4.21) converges to 0 as x goes to $\bar{x}$. This shows:

$$\mathbb{E}_{z,\eta}\left|\check{M}^V_t(x,u)-\check{M}^V_t(\bar{x},u)\right|^p\longrightarrow 0,\quad\text{as}\quad x\to\bar{x}\in\partial\mathcal{D}.$$

For any $p\ge 1$:

$$
\begin{aligned}
\mathbb{E}_{z,\eta}\left|\check{M}^Y_t(x,u)-\check{M}^Y_t(\bar{x},u)\right|^p&=\mathbb{E}_{z,\eta}\left|\int_0^t(\check{M}^V_s(x,u)-\check{M}^V_s(\bar{x},u))\,ds\right|^p\\
&\le t^{p-1}\int_0^t\mathbb{E}_{z,\eta}\left|\check{M}^V_s(x,u)-\check{M}^V_s(\bar{x},u)\right|^p\,ds
\end{aligned}
$$

and using Lebesgue convergence theorem and the previous convergence result, as $x\to\bar{x}$:

$$\mathbb{E}_{z,\eta}\left|\check{M}^Y_t(x,u)-\check{M}^Y_t(\bar{x},u)\right|^p\to 0.$$

Similar computations allow to show that for $x\to\bar{x}$:

$$
\begin{aligned}
&\mathbb{E}_{z,\eta}\left|\check{N}^Y_t(x,u)-\check{N}^Y_t(\bar{x},u)\right|^p\to 0\\
&\mathbb{E}_{z,\eta}\left|\check{N}^V_t(x,u)-\check{N}^V_t(\bar{x},u)\right|^p\to 0.
\end{aligned}
$$

□

Remark 4.4. *Following similar arguments as the ones presented in the proof of Lemma 4.3, we can show that for any $(t,x)\in[0,T]\times\overline{\mathcal{D}}$ and $p\in[1,\infty)$, the processes $\check{M}^Y_t(x,\cdot)$, $\check{M}^V_t(x,\cdot)$, $\check{N}^Y_t(x,\cdot)$ and $\check{N}^V_t(x,\cdot)$ are continuous on $\mathbb{R}^D$ in norm L^p.*

4.3 Application to the confined process

Girsanov transform

Consider the probability measure $\mathbb{P}_{x,u}$ defined by:

$$\left.\frac{d\mathbb{P}_{x,u}}{d\mathbb{P}_{z,\eta}}\right|_{\mathcal{F}_T}=G_T(x,u):=\exp\left(\int_0^T\widetilde{b}(z^{x,u}_s,\eta^u_s)\,d\check{W}_s-\frac{1}{2}\int_0^T\widetilde{b}^2(z^{x,u}_s,\eta^u_s)\,ds\right).\tag{4.23}$$

Since $\widetilde{b}$ is bounded, then, for any (x,u) in $\mathcal{D}\times\mathbb{R}^d$, $(G(x,u)_t)_{0\le t\le T}$ is a martingale and we have that $\mathbb{P}_{z,\eta}\sim\mathbb{P}_{x,u}$. By Girsanov's theorem, then the process $(z^{x,u}_t,\eta^u_t)_{0\le t\le T}$ solves the equation (4.1) under $\mathbb{P}_{x,u}$. This also means that (4.9) under $\mathbb{P}_{z,\eta}$ is equal in distribution to (4.6) under $\mathbb{P}_{x,u}$.

By (H_{PDE})-(i) and (H_{PDE})-(ii), we have that the function $\widetilde{b}$ is Lipschitz. Since the drift is sufficiently regular and σ is a constant, by [14], the stochastic flow process $(x,u)\mapsto(Y_t^{x,u},V_t^{x,u})$ is well defined and we can consider the function $f\colon\mathcal{D}\times\mathbb{R}^d\mapsto\mathbb{R}$ defined as $f(t,x,u):=\mathbb{E}_{x,u}\overline{\psi}(Y_t^{x,u},V_t^{x,u})$ where $\overline{\psi}$ is a continuous extension of the function ψ for negative values of $x^{(d)}$:

$$\overline{\psi}:(x,u)\in\mathbb{R}^d\times\mathbb{R}^d\mapsto\psi((x',|x^{(d)}|),(u',\operatorname{sign}(x^{(d)})u^{(d)})). \tag{4.24}$$

According to [5], the process defined as $(\mathfrak{X}_t,\mathfrak{U}_t)=((Y_t',|Y_t^{(d)}|),(V_t',(\operatorname{sign}(Y_t^{(d)})_+)\times V_t^{(d)})_{t\geq 0}$ is a weak solution of (1.1), so for any $(t,x,u)\in[0,T]\times\mathcal{D}\times\mathbb{R}^d$:

$$\begin{aligned}f(t,x,u)&=\mathbb{E}_{x,u}[\overline{\psi}(Y_t^{x,u},V_t^{x,u})]=\mathbb{E}_{x,u}[\mathbb{1}_{\{Y_t^{x,u}>0\}}\overline{\psi}(Y_t^{x,u},V_t^{x,u})]+\mathbb{E}_{x,u}[\mathbb{1}_{\{Y_t^{x,u}\leq 0\}}\overline{\psi}(Y_t^{x,u},V_t^{x,u})]\\&=\mathbb{E}_{x,u}[\mathbb{1}_{\{Y_t^{(d)x,u}>0\}}\psi(Y_t^{x,u},V_t^{x,u})]+\mathbb{E}_{x,u}\left[\mathbb{1}_{\{Y^{(d)}{}_t{}^{x,u}\leq 0\}}\psi\left((Y_t'^{x,u},-Y_t^{(d)x,u}),(V_t'^{x,u},-V_t^{(d)x,u}\right)\right]\\&=\mathbb{E}_{x,u}\left[\psi\left((Y_t'^{x,u},|Y_t^{(d)x,u}|),(V_t'^{x,u},\operatorname{sign}(Y_t^{(d)x,u})V_t^{(d)x,u}\right)\right]\\&=\mathbb{E}_{x,u}\left[\psi(\mathfrak{X}_t,\mathfrak{U}_t)\right]=\mathbb{E}_{x,u}[\psi(X_t^{x,u},U_t^{x,u})].\end{aligned} \tag{4.25}$$

We now state the lemma that contains a first part of the regularity results of Theorem 2.1:

Lemma 4.5. *The function F defined in (1.7) belongs in $\mathcal{C}([0,T];L^\infty(\mathcal{D}\times\mathbb{R}^d);\mathbb{R}^d)\cap\mathcal{C}([0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d;\mathbb{R})$.*

Proof. By similar arguments to (4.25), we can show that for any $(t,x,u)\in Q_T$, we have the equality

$$\mathbb{E}_{x,u}[\overline{\psi}(Y_T^{t,x,u},V_T^{t,x,u})]=\mathbb{E}_{x,u}[\psi(X_T^{t,x,u},U_T^{t,x,u})]$$

and they both equal $F(t,x,u)$ by definition (1.7). $(Y_T^{t,x,u},V_T^{t,x,u})$ is the solution at time T of the SDE (4.1) such that at time t, $(Y_t^{t,x,u},V_t^{t,x,u})=(x,u)$. By the hypotheses (H_{PDE})-(i) and (H_{PDE})-(ii), we have that the function $\widetilde{b}$ is Lipschitz (see Remark 1.4), therefore the flow $(t,x,u)\mapsto(Y_T^{t,x,u},V_T^{t,x,u})$ is almost surely continuous. The function ψ is continuous and bounded with support on $\mathcal{D}\times\mathbb{R}^d$, then $\overline{\psi}$ is also continuous and bounded. Let $(t,x,u)\in[0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d$ then for $(t_k,x_k,u_k)_{k\in\mathbb{N}}$ such that $(t_k,x_k,u_k)\to(t,x,u)$, when $k\to\infty$, we have that $\overline{\psi}(Y_T^{t_k,x_k,u_k},V_T^{t_k,x_k,u_k})\to\overline{\psi}(Y_T^{t,x,u},V_T^{t,x,u})$ a.s. Since $\overline{\psi}$ is bounded, then by the Dominated Convergence Theorem, $\mathbb{E}\left[\overline{\psi}(Y_T^{t_k,x_k,u_k},V_T^{t_k,x_k,u_k})\right]\to\mathbb{E}\left[\overline{\psi}(Y_T^{t,x,u},V_T^{t,x,u})\right]$, or written differently $F(t_k,u_k,x_k)\to F(t,x,u)$, when $(t_k,x_k,u_k)\to(t,x,u)$. This implies that F is continuous at $(t,x,u)\in[0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d$.

□

Lemma 4.6. *The function F defined in (1.7) is such that $\nabla_xF,\nabla_uF\in\mathcal{C}\left([0,T];L^\infty(\mathcal{D})\right)\cap\mathcal{C}([0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d)\cap L^2(Q_T,\mathbb{R}^d)\cap L^2(\Sigma_T,\mathbb{R}^d)$.*

Proof. Since for any $(t,x,u)\in Q_T$, $f(t,x,u)=\mathbb{E}_{x,u}[\psi(X_t^{x,u},U_t^{x,u})]$, $F(t,x,u)=\mathbb{E}[\psi(X_T^{t,x,u},U_T^{t,x,u})]$ and the fact that the process $(X_t,U_t)_{0\leq t\leq T}$ is time-homogeneous (as the drift b is not time dependent), then is clear that $f(t,x,u)=F(T-t,x,u)$. Therefore if the regularity results from the statement of the lemma are proven for f, they will also apply to F. We work with the former in this proof.

Provided sufficient regularity on $\overline{\psi}$, we have that:

$$\nabla_xf(t,x,u)=\mathbb{E}_{x,u}\left[\nabla_x\overline{\psi}(Y_t^{x,u},V_t^{x,u})M_t^Y(x,u)+\nabla_u\overline{\psi}(Y_t^{x,u},V_t^{x,u})M_t^V(x,u)\right]\quad\text{a.e.} \tag{4.26}$$

and

$$\nabla_uf(t,x,u)=\mathbb{E}_{x,u}\left[\nabla_x\overline{\psi}(Y_t^{x,u},V_t^{x,u})N_t^Y(x,u)+\nabla_u\overline{\psi}(Y_t^{x,u},V_t^{x,u})N_t^V(x,u)\right]\quad\text{a.e.} \tag{4.27}$$

By hypothesis (H_{PDE})-(i) and Gronwall's lemma, there exists a constant $C_{\nabla_xb,\nabla_ub,T}$ depending on ∇_xb, ∇_ub and T such that

$$\sup_{(t,x,u)\in\overline{Q_T}}\left(\left\|M_t^Y(x,u)\right\|+\left\|M_t^V(x,u)\right\|+\left\|N_t^Y(x,u)\right\|+\left\|N_t^V(x,u)\right\|\right)<C_{\nabla_xb,\nabla_ub,T}. \tag{4.28}$$

This result together with the boundedness of $\nabla_x\overline{\psi}$ and the bound (4.28) give that $\nabla_xf,\nabla_uf\in L^\infty(Q_t,\mathbb{R}^d)$.

Continuity of the derivatives

Let $(t,x,u)\in[0,T]\times\mathcal{D}\times\mathbb{R}^d$ and $\bar{x}$ the projection of x on $\partial\mathcal{D}$:

$$\begin{aligned}
&|\nabla_x f(t,x,u)-\nabla_x f(t,\bar{x},u)| = \left|\mathbb{E}_{x,u}\left[\nabla_x\overline{\psi}(Y_t^{x,u},V_t^{x,u})M_t^Y(x,u)+\nabla_u\overline{\psi}(Y_t^{x,u},V_t^{x,u})M_t^V(x,u)\right]\right.\\
&\quad\left.-\mathbb{E}_{\bar{x},u}\left[\nabla_x\overline{\psi}(Y_t^{\bar{x},u},V_t^{\bar{x},u})M_t^Y(\bar{x},u)+\nabla_u\overline{\psi}(Y_t^{\bar{x},u},V_t^{\bar{x},u})M_t^V(x,u)\right]\right|\\
&= \left|\mathbb{E}_{z,\eta}\left[G_t(x,u)\left(\nabla_x\overline{\psi}(z_t^{x,u},\eta_t^u)\check{M}_t^Y(x,u)+\nabla_u\overline{\psi}(z_t^{x,u},\eta_t^u)\check{M}_t^V(x,u)\right)\right]\right.\\
&\quad\left.-\mathbb{E}_{z,\eta}\left[G_t(\bar{x},u)\left(\nabla_x\overline{\psi}(z_t^{\bar{x},u},\eta_t^u)\check{M}_t^Y(\bar{x},u)+\nabla_u\overline{\psi}(z_t^{\bar{x},u},\eta_t^u)\check{M}_t^V(\bar{x},u)\right)\right]\right|\\
&\le \mathbb{E}_{z,\eta}\left|G_t(x,u)\nabla_x\overline{\psi}(z_t^{x,u},\eta_t^u)\check{M}_t^Y(x,u)-G_t(\bar{x},u)\nabla_x\overline{\psi}(z_t^{\bar{x},u},\eta_t^u)\check{M}_t^Y(\bar{x},u)\right|\\
&\quad+\mathbb{E}_{z,\eta}\left|G_t(x,u)\nabla_u\overline{\psi}(z_t^{x,u},\eta_t^u)\check{M}_t^V(x,u)-G_t(\bar{x},u)\nabla_u\overline{\psi}(z_t^{\bar{x},u},\eta_t^u)\check{M}_t^V(\bar{x},u)\right|.
\end{aligned}\tag{4.29}$$

Considering the first term:

$$\begin{aligned}
&\mathbb{E}_{z,\eta}\left|G_t(x,u)\nabla_x\overline{\psi}(z_t^{x,u},\eta_t^u)\check{M}_t^Y(x,u)-G_t(\bar{x},u)\nabla_x\overline{\psi}(z_t^{\bar{x},u},\eta_t^u)\check{M}_t^Y(\bar{x},u)\right|\\
&\le \mathbb{E}_{z,\eta}\left|G_t(x,u)-G_t(\bar{x},u)\right|\left|\nabla_x\overline{\psi}(z_t^{x,u},\eta_t^u)\check{M}_t^Y(x,u)\right|\\
&\quad+\mathbb{E}_{z,\eta}G_t(\bar{x},u)\left|\check{M}_t^Y(x,u)\right|\nabla_x\overline{\psi}(z_t^{x,u},\eta_t^u)-\nabla_x\overline{\psi}(z_t^{\bar{x},u},\eta_t^u)\,||\\
&\quad+\mathbb{E}_{z,\eta}G_t(\bar{x},u)\left|\nabla_x\overline{\psi}(z_t^{\bar{x},u},\eta_t^u)\right|\left|\check{M}_t^Y(x,u)-\check{M}_t^Y(\bar{x},u)\right|\\
&\le \left\|\nabla_x\overline{\psi}\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^d)}\left(\sup_{t,x,u\in Q_T}|\check{M}_t^Y(x,u)|\right)\mathbb{E}_{z,\eta}\left|G_t(x,u)-G_t(\bar{x},u)\right|\\
&\quad+\left(\sup_{t,x,u\in Q_T}|\check{M}_t^Y(x,u)|\right)\left(\mathbb{E}_{z,\eta}G_t(\bar{x},u)^2\right)^{\frac12}\left(\mathbb{E}_{z,\eta}\left|\nabla_x\overline{\psi}(z_t^{x,u},\eta_t^u)-\nabla_x\overline{\psi}(z_t^{\bar{x},u},\eta_t^u)\right|^2\right)^{\frac12}\\
&\quad+\left\|\nabla_x\overline{\psi}\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^d)}\left(\mathbb{E}_{z,\eta}G_t(\bar{x},u)^2\right)^{\frac12}\left(\mathbb{E}_{z,\eta}\left|\nabla_x\overline{\psi}(z_t^{\bar{x},u},\eta_t^u)\right|\left|\check{M}_t^Y(x,u)-\check{M}_t^Y(\bar{x},u)\right|^2\right)^{\frac12}.
\end{aligned}$$

Since the function $\widetilde{b}$ is Lipschitz by the hypotheses (H_{PDE})-(i) and (H_{PDE})-(ii), (see Remark 1.4) and, by their definitions in (4.8), for any $(t,u)\in[0,T]\times\mathbb{R}^d$, the function $x\mapsto(z_t^{x,u},\eta_t^u)$ is continuous. Then a.s. the function $x\mapsto G_t(x,u)$ for any $u\in\mathbb{R}^d$ is also continuous. For all $x\in\mathcal{D}$, $\mathbb{E}_{z,\eta}G_t(x,u)=\mathbb{E}_{z,\eta}G_t(\bar{x},u)=1$, then the first term of the inequality goes to 0 as $x\to\bar{x}$ by Lebesgue Dominated Convergence theorem. Similarly, the Lipschitz continuity of $\partial_x\overline{\psi}$ and the L^p continuity of $(z_t^{x,u})_{0\le t\le T}$ in its initial condition implies that the second term converges to 0 as $x\to\bar{x}$. Also lemma 4.3 shows that the third term of the sum also goes to 0.

Similar arguments show that the second term of the bound in (4.29) goes to 0 as $x\to\bar{x}$, thus proving that $\nabla_x f(t,\cdot,u)$ is continuous up to the border $\partial\mathcal{D}$. And by repeating the same arguments, only replacing $\check{M}_t^Y(\cdot,u)$ and $\check{M}_t^V(\cdot,u)$ with $\check{N}_t^Y(\cdot,u)$ and $\check{N}_t^V(\cdot,u)$, we obtain also that $\nabla_u f(t,\cdot,u)$ is continuous up to the border $\partial\mathcal{D}$.

Through an analogous procedure that involves the continuity of $G_t(x,\cdot)$, the L^p continuity as expressed in the Remark 4.4 and boundedness on $\overline{Q_T}$ shown in (4.28) of $\check{M}_t^Y(x,\cdot)$, $\check{M}_t^V(x,\cdot)$, $\check{N}_t^Y(x,\cdot)$ and $\check{N}_t^V(x,\cdot)$, it can be shown that the functions $\nabla_x f(t,x,\cdot)$ and $\nabla_u f(t,x,\cdot)$ are continuous for any $(t,x)\in[0,T]\times\overline{\mathcal{D}}$ and the same for $\nabla_x f(\cdot,x,u)$ and $\nabla_u f(\cdot,x,u)$ for any $(x,u)\in\overline{\mathcal{D}}\times\mathbb{R}^d$.

Existence of the L^2 norms

Let $(t,x,u)\in\overline{Q_t}$, then

$$\begin{aligned}
|\nabla_x f(t,x,u)| &\le C_{\check{M}^Y,\check{M}^V}\left(\mathbb{E}_{z,\eta}G_t(x,u)^2\right)^{\frac12}\left(\mathbb{E}_{z,\eta}\left(\left|\partial_x\overline{\psi}(z_t^{x,u},\eta_t^u)\right|+\left|\partial_u\overline{\psi}(z_t^{x,u},\eta_t^u)\right|\right)^2\right)^{\frac12}\\
&\le C_{\check{M}^Y,\check{M}^V}e^{\frac74 T\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d)}}\left(\mathbb{E}_{z,\eta}\left(\left|\partial_x\overline{\psi}(z_t^{x,u},\eta_t^u)\right|+\left|\partial_u\overline{\psi}(z_t^{x,u},\eta_t^u)\right|\right)^2\right)^{\frac12}
\end{aligned}$$

where $C_{\check{M}^Y,\check{M}^V}=\max\left\{\left\|\check{M}^Y\right\|_{L^\infty(Q_T,\mathbb{R}^{2d})},\left\|\check{M}^V\right\|_{L^\infty(Q_T,\mathbb{R}^{2d})}\right\}$. Since $\overline{\psi}\in\mathcal{C}_c^{1,1}(\mathbb{R}^d\times\mathbb{R}^d)$, then we also have that $\nabla_x\overline{\psi}\in\mathcal{C}_c^{0,1}(\mathbb{R}^d\times\mathbb{R}^d)$ and $\nabla_u\overline{\psi}\in\mathcal{C}_c^{0,1}(\mathbb{R}^d\times\mathbb{R}^d)$. So there exists two non-negative function $\beta_1,\beta_2:\mathbb{R}^d\mapsto\mathbb{R}$ such that $\beta_1(x)=1$ for $x\in\mathrm{Proj}_x\left(\mathrm{Supp}(\overline{\psi})\right)$ and 0 everywhere else, and $\beta_2(u)=1$ for $u\in\mathrm{Proj}_u\left(\mathrm{Supp}(\overline{\psi})\right)$ and 0 everywhere else (where Proj_x and Proj_u are the projections according to the first d and the last d dimensions of $\mathbb{R}^{d\times d}$) and a constant

$$C=\sup_{(x,u)\in\mathcal{D}\times\mathbb{R}^d}\left(\left|\nabla_x\overline{\psi}(x,u)\right|+\left|\nabla_u\overline{\psi}(x,u)\right|\right)^2$$

such that

$$\left(\left|\nabla_x\overline{\psi}(x,u)\right| + \left|\nabla_u\overline{\psi}(x,u)\right|\right)^2 \le C\beta_1(x)\beta_2(u) \implies |\nabla_x f(t,x,u)| \le C_{\check{M},T,b}\left(\mathbb{E}_{z,\eta}\beta_1(z_t^{x,u})\beta_2(\eta_t^u)\right)^{\frac{1}{2}}$$

where $C_{\check{M},T,b} = CC_{\check{M}^Y,\check{M}^V}e^{\frac{7}{4}T\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d)}}$, so we can rewrite:

$$|\nabla_x f(t,x,u)| \le C_{\check{M},T,b}\left(\int_{\mathbb{R}^{2d}}\beta_1(x+ut+z)\beta_2(u+\eta)\check{p}(t;z,\eta)\,dzd\eta\right)^{\frac{1}{2}}$$

the function $\check{p}(t;\cdot,\cdot)$ being the density of the random variable $\left(\sigma\int_0^t \check{W}_s\,ds, \sigma\check{W}_t\right)$. Finally

$$\begin{aligned}
\|\nabla_x f\|^2_{L^2(\Sigma_T,\mathbb{R}^d)} &= \int_{\Sigma_T}|(u\cdot n_{\mathcal{D}}(x))|\,|\nabla_x f(t,x,u)|^2\,dt\otimes d\sigma_{\partial\mathcal{D}}(x)\otimes du\\
&\le C^2_{\check{M},T,b}\int_0^T\int_\Sigma\int_{\mathbb{R}^{2d}}|u|\beta_1(x+ut+z)\beta_2(u+\eta)\check{p}(t;z,\eta)\,dzd\eta dt\otimes d\sigma_{\partial\mathcal{D}}(x)\otimes du\\
&\le C^2_{\check{M},T,b}\int_0^T dt\int_{\mathbb{R}^{2d}}\check{p}(t;z,\eta)\,dzd\eta\int_{\mathbb{R}^d}|u|\beta_2(u+\eta)\,du\int_{\partial\mathcal{D}}\beta_1(x+ut+z)\,d\sigma_{\partial\mathcal{D}}(x)\\
&\le C^2_{\check{M},T,b}\lambda(\partial\mathcal{D}\cap\mathrm{Supp}(\overline{\psi}))\int_0^T dt\int_{\mathbb{R}^d}\check{p}(t;z,\eta)\,dzd\eta\left(\int_{\mathbb{R}^d}(|u+\eta|+|\eta|)\beta_2(u+\eta)\,du\right)\\
&\le C^2_{\check{M},T,b}\lambda(\partial\mathcal{D}\cap\mathrm{Supp}(\overline{\psi}))\int_0^T dt\int_{\mathbb{R}^d}\frac{1}{(\sigma\sqrt{t})^d}p^{(d)}_{\mathcal{N}(0,I_d)}\left(\frac{\eta}{\sigma\sqrt{t}}\right)d\eta\left(C_{\beta_2}+|\eta|\lambda(\mathrm{Supp}(\overline{\psi}))\right)
\end{aligned}\tag{4.30}$$

where $C_{\beta_2} = \int_{\mathbb{R}^d}|u|\beta_2(u)\,du$ is a constant and $p^{(d)}_{\mathcal{N}(0,I_d)}$ is the density of the centred $d-$dimensional normal distribution that has for covariance matrix I_d which admits moments of any order so the double integral left in the final equality is finite. Similar computations show that $\|\nabla_u f\|_{L^2(\Sigma_T)}$ is finite.

Now, we consider the norm:

$$\begin{aligned}
\|\nabla_x f\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)} &= \int_{\mathcal{D}\times\mathbb{R}^d}|\nabla_x f(t,x,u)|^2\,dtdxdu\\
&\le C^2_{\check{M},T,b}\int_{\mathcal{D}}\int_{\mathbb{R}^{2d}}\beta_1(x+ut+z)\beta_2(u+\eta)\check{p}(t;z,\eta)\,dzd\eta dxdu\\
&\le C^2_{\check{M},T,b}\int_{\mathbb{R}^{2d}}\check{p}(t;z,\eta)\,dzd\eta\int_{\mathbb{R}^d}\beta_2(u+\eta)\,du\int_{\mathcal{D}}\beta_1(x+ut+z)\,dx\\
&\le C^2_{\check{M},T,b}\lambda\left(\mathrm{Supp}(\overline{\psi})\right)^2.
\end{aligned}$$

Corollary 6.3 gives the result that $\nabla_u f \in L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)$. □

5 Regularity of the Kolmogorov problem with specular boundary conditions

The bounds of the weak error (3.46) obtained in section 3 also depend on the $L^2(Q_T;\mathbb{R}^{2d})$ norms of $\mathrm{Hess}_{x,u}(F)$ and $\mathrm{Hess}_{u,u}(F)$ where F is the solution in distribution of (1.9), or under a probabilistic interpretation (1.7). This section focuses on this L^2 regularity of these second order derivatives, which is the final result of Theorem (2.1). Instead of working on this function, we consider the following $f\colon Q_T\mapsto\mathbb{R}$, $f(t,x,u) = \mathbb{E}\psi\left(X_t^{x,u},U_t^{x,u}\right)$ for any $(t,x,u)\in Q_T$. As mentioned in the previous section, $f(t,x,u) = F(T-t,x,u)$, so the $L^2(Q_T,\mathbb{R}^{2d})$ regularity of the second order derivatives proven for one function, apply to the other. Again, we consider the former which verifies the equation (6.11).

Remark 5.1. *The solution f of (6.11) is in distribution, meaning that for any $\varphi \in \mathcal{C}_b^\infty(\overline{Q}_t)$ we have:*

$$\begin{aligned}
&\int_{Q_t} f(s,x,u)\left(\partial_s\varphi - (u\cdot\nabla_x\varphi) - (\nabla_u\cdot(b(x,u)\varphi)) + \frac{\sigma^2}{2}\triangle_u\varphi\right)(s,x,u)\,ds\,dx\,du \\
&= \int_{\mathcal{D}\times\mathbb{R}^d}\left[\varphi(s,x,u)f(s,x,u)\right]_{s=0}^{s=t}dx\,du - \int_{\Sigma_t^-}(u\cdot n_{\mathcal{D}}(x))\gamma(f)(s,x,u)\varphi(s,x,u)\,d\lambda_{\Sigma_T}(s,x,u) \\
&\quad - \int_{\Sigma_t^+}(u\cdot n_{\mathcal{D}}(x))\gamma(f)(s,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))\varphi(s,x,u)\,d\lambda_{\Sigma_T}(s,x,u),
\end{aligned} \tag{5.1}$$

Let us further notice that the trace function $\gamma(f)$ in $L^2(\Sigma_T)$ is characterized by the Green formula related to the transport operator $\partial_t + (u\cdot\nabla_x)$ (we refer to Subsection C.1 for more details).

We extend the mollifiers defined in the introduction of Section 3 for $d \geq 2$. Let $(\widetilde{\beta}_k)_{k\geq1}$, $(\rho_n)_{n\geq1}$ and $(g_m)_{m\geq1}$ be smooth sequences such that:

$$\operatorname{Supp}(\widetilde{\beta}_k) \subset \left(-\frac{T}{k},0\right), \qquad \operatorname{Supp}(\rho_n) \subset \mathcal{B}_{\frac{1}{n}}(0;\mathbb{R}^{d-1}) \times \left(-\frac{1}{n},0\right) \qquad \text{and} \qquad \operatorname{Supp}(g_m) = \mathbb{R}^d \tag{5.2}$$

where $\mathcal{B}_r\left(a;\mathbb{R}^{d-1}\right)$ is the $\mathbb{R}^{d-1}$ open ball centered at $a \in \mathbb{R}^{d-1}$ with radius r

The sequence $(\widetilde{\beta}_k)_{k\geq1}$ is defined using the mollifying sequence $(\beta_k)_{k\geq1}$ from Section 3. For any $t \in \mathbb{R}$, we state that $\widetilde{\beta}_k(t) = \beta_k(-t)$. So $\widetilde{\beta}_k$ is reflection according to the abscissa of β_k.

Recalling the notation $x = (x', x^{(d)})$, for any $x \in \mathbb{R}^d$, we consider the generating function:

$$\rho\colon x \mapsto \begin{cases} \exp\left(-\dfrac{1}{1-\|x'\|^2}\right)\exp\left(-\dfrac{1}{x^{(d)}(-1-x^{(d)})}\right) & \text{for } x \in \mathcal{B}_1(0;\mathbb{R}^{d-1})\times(-1,0)\,, \\ 0 \quad \text{otherwise}\,, \end{cases}$$

then for any $n \geq 1$, $\rho_n(x) = Cn^d\rho(nx)$ where C is such that $\displaystyle\int_{\mathbb{R}^d}\rho(x)\,dx = \frac{1}{C}$.

For the sequence $(g_m)_{m\geq1}$ we choose to use the Gaussian kernel:

$$g\colon u \in \mathbb{R} \mapsto \frac{1}{\sqrt{(2\pi)^d}}\exp\left(-\frac{\|u\|^2}{2}\right)$$

by taking $g_m(u) = m^d g(mu)$, with the property that

$$ug_m(u) = -\frac{1}{m^2}\nabla_u g_m(u). \tag{5.3}$$

We define the regularisation of f the solution in distribution of (6.11) as $f_{k,n,m}\colon (\tau,y,v) \in \overline{Q_T} \mapsto \mathbb{R}$ as

$$f_{k,n,m}(\tau,y,v) = \int_{Q_T} f(s,x,u)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu. \tag{5.4}$$

Also defined is $f_{n,m}$, the regularisation of f only w.r.t. the spatial coordinates, defined for every $(s,y,v) \in \overline{Q_T}$ as:

$$f_{n,m}(s,y,v) = \int_{\mathcal{D}\times\mathbb{R}^d} f(s,x,u)\rho_n(y-x)g_m(v-u)\,dxdu\,. \tag{5.5}$$

In the following Lemma, we obtain the equality verified by $f_{k,n,m}$.

Lemma 5.2. *The function $f_{\delta,n,m}$ on the interior of Q_T satisfies the equality*

$$\begin{aligned}
&-\partial_\tau f_{k,n,m}(\tau,y,v) + (v\cdot\nabla_y f_{k,n,m})(\tau,y,v) + (b(y,v)\cdot\nabla_v f_{k,n,m}))(\tau,y,v) + \frac{\sigma^2}{2}\triangle_v f_{k,n,m}(\tau,y,v) \\
&= R_{k,n,m}[f](\tau,y,v).
\end{aligned} \tag{5.6}$$

with

$$R_{k,n,m}[f](\tau,y,v) = R^{\mathrm{Sp}}_{k,n,m}[f](\tau,y,v) + R^{\mathrm{Tm}}_{k,n,m}[f](\tau,y,v)$$

where

$$\begin{aligned} R^{\mathrm{Sp}}_{k,n,m}[f](\tau,y,v) &:= f * \left(\widetilde{\beta}_k(\nabla_y \rho_n \cdot (v g_m))\right) \\ &\qquad + \int_{Q_T} ((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u))\,\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu \\ R^{\mathrm{Tm}}_{k,n,m}[f](\tau,y,v) &:= \widetilde{\beta}_k(\tau-T) f_{n,m}(T,y,v)\,. \end{aligned} \tag{5.7}$$

Proof. To prove this Lemma, we consider a specific test function that is applied to the equation in Remark 5.1 and which gives the desired result.

We consider a test function $\varphi \in \mathcal{C}^\infty_b(\overline{Q}_T)$. The function $\hat{\varphi}_{k,n,m}\colon (s,x,u)\in Q_T \mapsto \hat{\varphi}_{k,n,m}(s,x,u)\in\mathbb{R}$ defined as:

$$\hat{\varphi}_{k,n,m}(s,x,u) = \int_{Q_T} \varphi(\tau,y,v)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,d\tau dydv$$

is in $\mathcal{C}^\infty_b(\overline{Q}_T)$ and $\hat{\varphi}_{k,n,m}$ vanished close to $\partial\mathcal{D}$ since the support of $\rho_n(y-\cdot)$ is in $\mathcal{D}$ for any $y\in\mathcal{D}$. We mention that the mollifying sequence $(\rho_n)_{n\geq 1}$ has been chosen such that it removes the contribution of the boundary Σ_T in the equation (5.1). The Remark 5.1 applies for the test function $\hat{\varphi}_{k,n,m}(t,x,u)$ and on the whole domain Q_T we obtain:

$$\begin{aligned} &\int_{Q_T} f(s,x,u)\left(\partial_s\hat{\varphi}_{k,n,m} - (u\cdot\nabla_x\hat{\varphi}_{k,n,m}) - \nabla_u\cdot(b(x,u)\hat{\varphi}_{k,n,m}) + \frac{\sigma^2}{2}\triangle_u\hat{\varphi}_{k,n,m}\right)(s,x,u)\,dsdxdu \\ &= \int_{\mathcal{D}\times\mathbb{R}^d} \left[\hat{\varphi}_{k,n,m}(s,x,u)f(s,x,u)\right]_{s=0}^{s=T} dxdu. \end{aligned} \tag{5.8}$$

By using Fubini's theorem, we pass the mollifiers on the function f in order to obtain the equality for function $f_{k,n,m}$.

We start by analysing every term of equation (5.8), one by one. The first term corresponds to the derivative in time, and by noticing that $\partial_s\widetilde{\beta}_k(\tau-s) = -\partial_\tau\widetilde{\beta}_k(\tau-s)$:

$$\begin{aligned} &\int_{Q_T} f(s,x,u)\partial_s\hat{\varphi}_{k,n,m}(s,x,u)\,dsdxdu \\ &= \int_{Q_T} f(s,x,u)\partial_s\left(\int_{Q_T}\varphi(\tau,y,v)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,d\tau dydv\right) dsdxdu \\ &= \int_{Q_T\times Q_T} f(s,x,u)\varphi(\tau,y,v)\partial_s\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,d\tau dydv\,dsdxdu \\ &= -\int_{Q_T\times Q_T} f(s,x,u)\varphi(\tau,y,v)\partial_\tau\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,d\tau dydv\,dsdxdu \\ &= -\int_{Q_T}\varphi(\tau,y,v)\partial_\tau\left(\int_{Q_T} f(s,x,u)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right) d\tau dydv \\ &= -\int_{Q_T}\varphi(\tau,y,v)\partial_\tau f_{k,n,m}(\tau,y,v)\,d\tau dydv \end{aligned}$$

while the r.h.s term of equation (5.8) becomes:

$$
\begin{aligned}
&\int_{\mathcal{D}\times\mathbb{R}^d} \left[\hat{\varphi}_{k,n,m}(s,x,u)f(s,x,u)\right]_{s=0}^{s=T} dxdu \\
&= \int_{\mathcal{D}\times\mathbb{R}^d} \hat{\varphi}_{k,n,m}(T,x,u)f(T,x,u)\,dxdu - \int_{\mathcal{D}\times\mathbb{R}^d} \hat{\varphi}_{k,n,m}(0,x,u)f(0,x,u)\,dxdu \\
&= \int_{Q_T} \varphi(\tau,y,v)\widetilde{\beta}_k(\tau-T)\int_{\mathcal{D}\times\mathbb{R}^d} f(T,x,u)\rho_n(y-x)g_m(v-u)\,dxdud\tau dydv \\
&- \int_{Q_T} \varphi(\tau,y,v)\widetilde{\beta}_k(\tau)\int_{\mathcal{D}\times\mathbb{R}^d} f(0,x,u)\rho_n(y-x)g_m(v-u)\,dxdud\tau dydv \\
&= \int_{Q_T} \varphi(\tau,y,v)\widetilde{\beta}_k(\tau-T)f_{n,m}(T,y,v)\,d\tau dydv - \int_{Q_T} \varphi(\tau,y,v)\widetilde{\beta}_k(\tau)f_{n,m}(0,y,v)\,d\tau dydv \\
&= \int_{Q_T} \varphi(\tau,y,v)\left[\widetilde{\beta}_k(\tau-s)f_{n,m}(s,y,v)\right]_{s=0}^{s=T} d\tau dydv
\end{aligned}
\tag{5.9}
$$

The term $s=0$ is zero since the support of $\widetilde{\beta}_k$ is included just on $[-T,0]$.

By Fubini's theorem, the term corresponding to the drift in x in equation (5.8) can be rewritten as:

$$
\begin{aligned}
&\int_{Q_T} f(s,x,u)(u\cdot\nabla_x\hat{\varphi}_{k,n,m}(s,x,u))\,dsdxdu \\
&= \int_{[0,T]^2\times\mathbb{R}^{2d}} \widetilde{\beta}_k(\tau-s)g_m(v-u)\,dsd\tau dudv \int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)(u\cdot\nabla_x\rho(y-x))\,dxdy
\end{aligned}
\tag{5.10}
$$

and, for the sake of simplicity we develop just the inner integral. Since $\nabla_x\rho(y-x) = -\nabla_y\rho(y-x)$, we have that

$$
\begin{aligned}
&\int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)(u\cdot\nabla_x\rho_n(y-x))\,dxdy = -\int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)(u\cdot\nabla_y\rho_n(y-x))\,dxdy \\
&= -\int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)(v\cdot\nabla_y\rho_n(y-x))\,dxdy + \int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)((v-u)\cdot\nabla_y\rho_n(y-x))\,dxdy \\
&= -\int_{\mathcal{D}} \varphi(\tau,y,v)\left(v\cdot\nabla_y\int_{\mathcal{D}} f(s,x,u)\rho_n(y-x)\,dx\right)dy + \int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)((v-u)\cdot\nabla_y\rho_n(y-x))\,dxdy
\end{aligned}
\tag{5.11}
$$

This means that we can rewrite the term (5.10) as:

$$
\begin{aligned}
&\int_{Q_T} f(s,x,u)(u\cdot\nabla_x\hat{\varphi}_{k,n,m}(s,x,u))\,dsdxdu = -\int_{Q_T} \varphi(\tau,y,v)\left(v\cdot\nabla_y f_{k,n,m}\right)d\tau dydv \\
&+ \int_{Q_T} \varphi(\tau,y,v)\left(\int_{Q_T} f(s,x,u)((v-u)\cdot\nabla_y\rho_n(y-x))\widetilde{\beta}_k(\tau-s)g_m(v-u)\,dsdxdu\right)d\tau dydv
\end{aligned}
\tag{5.12}
$$

For the term corresponding to the drift in u in equation (5.8), we can perform an i.b.p. because $f\in H^1(\mathbb{R}^d)$ in the variable u according to Lemma (4.6), we then add and subtract a term in $b(y,v)$, and perform and integration by parts on one of these terms to obtain:

$$
\begin{aligned}
&\int_{Q_T} f(s,x,u)(\nabla_u\cdot(b(x,u)\hat{\varphi}_{k,n,m}(s,x,u)))\,dsdxdu = \int_{Q_T} \hat{\varphi}_{k,n,m}(s,x,u)(b(x,u)\cdot\nabla_u f(s,x,u))\,dsdxdu \\
&= \int_{Q_T^2} f(s,x,u)\varphi(\tau,y,v)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)(b(y,v)\cdot\nabla_u g_m(v-u))\,dsd\tau dxdydudv \\
&- \int_{Q_T^2} (\nabla_u f(s,x,u)\cdot((b(x,u)-b(y,v)))\varphi(\tau,y,v)\rho_n(y-x)g_m(v-u)\widetilde{\beta}_k(\tau-s)\,dsd\tau dxdydudv.
\end{aligned}
\tag{5.13}
$$

As $\nabla_u g_m(v-u) = -\nabla_v g_m(v-u)$, we have, after several applications of Fubini's theorem:

$$
\begin{aligned}
&\int_{Q_T} f(s,x,u)(\nabla_u \cdot (b(x,u)\hat{\varphi}_{k,n,m}))\, dsdxdu \\
&= -\int_{Q_T} \varphi(\tau,y,v)\,(b(y,v)\cdot\nabla_v f_{k,n,m})\; dsdydv \\
&\quad -\int_{Q_T} \varphi(\tau,y,v)\left(\int_{Q_T} ((b(x,u)-b(y,v))\cdot\nabla_u f(s,x,u))\,\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\, dsdxdu\right) d\tau dydv
\end{aligned}
\tag{5.14}
$$

The diffusion term becomes

$$
\frac{\sigma^2}{2}\int_{Q_T} f(s,x,u)\triangle_u\hat{\varphi}_{k,n,m}(s,x,u)\, dsdxdu = \frac{\sigma^2}{2}\int_{Q_T} \varphi(\tau,y,v)\triangle_v f_{k,n,m}(\tau,y,v)\, d\tau dydv \tag{5.15}
$$

The smoothed version $f_{k,n,m}$ on Q_T of the weak solution f of (5.1) verifies for any $\varphi\in C_b^\infty(\overline{Q}_T)$:

$$
\begin{aligned}
&-\int_{Q_T} \varphi(\tau,y,v)\partial_\tau f_{k,n,m}(\tau,y,v)\, d\tau dydv + \int_{Q_T} \varphi(\tau,y,v)\,(v\cdot\nabla_y f_{k,n,m})\; d\tau dydv \\
&\quad + \int_{Q_T} \varphi(\tau,y,v)\,(b(y,v)\cdot\nabla_v f_{k,n,m})\; dsdydv + \frac{\sigma^2}{2}\int_{Q_T} \varphi(\tau,y,v)\triangle_v f_{k,n,m}(\tau,y,v)\, d\tau dydv \\
&= \int_{Q_T} \varphi(\tau,y,v)\big[\widetilde{\beta}_k(\tau-s)f_{n,m}(s,y,v)\big]_{s=0}^{s=T}\; d\tau dydv \\
&\quad + \int_{Q_T} \varphi(\tau,y,v)\left(\int_{Q_T} f(s,x,u)((v-u)\cdot\nabla_y\rho_n(y-x))\widetilde{\beta}_k(\tau-s)g_m(v-u)\, dsdxdu\right) d\tau dydv \\
&\quad + \int_{Q_T} \varphi(\tau,y,v)\left(\int_{Q_T} ((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u))\,\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\, dsdxdu\right) d\tau dydv
\end{aligned}
\tag{5.16}
$$

where $f_{n,m}$ is defined in (5.5).

We have that $\mathrm{Supp}(\widetilde{\beta}_k)\subset[-T,0]$ so for any $\tau\in[0,T]$, $\widetilde{\beta}_k(t)=0$ and since $f_{k,n,m}$ is a smooth function in the interior of Q_T, we obtain that

$$
\begin{aligned}
&-\partial_\tau f_{k,n,m}(\tau,y,v) + (v\cdot\nabla_y f_{k,n,m})(\tau,y,v) + (b(y,v)\cdot\nabla_v f_{k,n,m})\,(\tau,y,v) + \frac{\sigma^2}{2}\triangle_v f_{k,n,m}(\tau,y,v) \\
&= R_{k,n,m}[f](\tau,y,v).
\end{aligned}
\tag{5.17}
$$

□

Lemma 5.3. *Consider a function f such that $f,\nabla_u f,\nabla_x f \in \mathcal{C}([0,T];L^\infty(\mathcal{D}))\cap\mathcal{C}([0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d)\cap L^2(Q_T;\mathbb{R}^d)\cap L^2(\Sigma_T,\mathbb{R}^d)$ and define for any $(\tau,y,v)\in Q_T$*

$$
\begin{aligned}
&R^{\mathrm{Sp}}_{k,n,m}[f](\tau,y,v) \\
&:= f*\Big(\widetilde{\beta}_k(\nabla_y\rho_n\cdot(vg_m))\Big) + \int_{Q_T} ((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u))\,\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\, dsdxdu
\end{aligned}
\tag{5.18}
$$

and

$$
R^{\mathrm{Tm}}_{k,n,m}[f](\tau,y,v) := \widetilde{\beta}_k(\tau-T)f_{n,m}(T,y,v) \tag{5.19}
$$

By considering $n\sim m$ at infinity, then:

i) $\left\|R^{\mathrm{Sp}}_{k,n,m}[f](\tau,y,v)\right\|_{L^\infty(Q_T)} \xrightarrow[(k,n,m)\to\infty]{} 0$

ii) $\left\|\nabla_y R^{\mathrm{Sp}}_{k,n,m}[f](\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)} \le C_{\nabla_u f,b}$ *and* $\left\|\nabla_y R^{\mathrm{Sp}}_{k,n,m}[f]\right\|_{L^2(Q_T;\mathbb{R}^d)} \le C_{\nabla_u f,b}$*, uniformly in (k,n,m)*

iii) $\left\|\nabla_v R^{\text{Sp}}_{k,n,m}[f](\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)} \leq C_{\nabla_u f,b}$ *and* $\left\|\nabla_v R^{\text{Sp}}_{k,n,m}[f]\right\|_{L^2(Q_T;\mathbb{R}^d)} \leq C_{\nabla_u f,b}$ *uniformly in* (k,n,m)

iv) $\left|f(T,y,v) - \int_0^T R^{\text{Tm}}_{k,n,m}[f](\tau,y,v)\,d\tau\right| \xrightarrow[(n,m)\to\infty]{} 0$

v) $\int_0^T \left\|\nabla_u R^{\text{Tm}}_{k,n,m}[f](\tau,y,v)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \leq C$ *and* $\int_0^T \left\|\nabla_x R^{\text{Tm}}_{k,n,m}[f](\tau,y,v)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \leq C$ *uniformly in* (k,l,m).

Proof.

For the proof of this Lemma we utilise several properties on the mollifiers $(\rho_n)_{n\geq 1}$ and $(g_m)_{m\geq 1}$ defined at the beginning of this section. We have that

$$\int_{\mathbb{R}^d} \|x\|\,\rho_n(x)\,dx = \int_{\text{Supp}(\rho_n)} \|x\|\,\rho_n(x)\,dx \leq \frac{1}{n}\int_{\text{Supp}(\rho_n)} \rho_n(x)\,dx = \frac{1}{n}$$

then

$$\begin{aligned}\int_{\mathbb{R}^d} \left|\text{Hess}_{x,x}(\rho_n)(x)\right|\,dx &= Cn^d\int_{\mathbb{R}^d}\left|\text{Hess}_{x,x}(\rho(nx))\right|\,dx\\ = Cn^2n^d\int_{\mathbb{R}^d}\left|\text{Hess}_{x,x}(\rho)(nx)\right|\,dx &= Cn^2\int_{\mathbb{R}^d}\left|\text{Hess}_{x,x}(\rho)(x)\right|\,dx \leq C_{\nabla^2_x\rho_1}n^2\end{aligned}$$

where $C_{\nabla^2_x\rho_1}$ depends on C the integral of ρ and on the integral of the Hessian of ρ. Finally we have that

$$\int_{\mathbb{R}^d}\|x\|\left|\nabla_x\rho_n(x)\right|\,dx \leq \frac{1}{n}n^d n\int_{\mathbb{R}^d}\left|\nabla_x\rho(nx)\right|\,dx \leq \int_{\mathbb{R}^d}\left|\nabla_x\rho(x)\right|\,dx$$

Similar properties are deduced for $(g_m)_{m\geq 1}$.

i) **Convergence of the error.**

We consider the first term of $R^{\text{Sp}}_{k,n,m}[f]$ and the property (5.3):

$$\begin{aligned}\left|f * \left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot(vg_m))\right)(\tau,y,v)\right| &= \frac{1}{m^2}\left|f * \left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot(\nabla_v g_m))\right)(\tau,y,v)\right|\\ &= \frac{1}{m^2}\left|\nabla_v f * \left(\widetilde{\beta}_k\nabla_y\rho_n g_m\right)(\tau,y,v)\right|\\ &\leq \frac{n}{m^2}\left\|\nabla_v f\right\|_{L^\infty(Q_T;\mathbb{R}^d)}C_{\nabla_x\rho_1}\end{aligned} \tag{5.20}$$

where $C_{\nabla_x\rho_1}$ depends only on the gradient of ρ_1.

Since the function b is Lipschitz by hypothesis ($H_{Langevin}$)-(ii) and $\text{Supp}(\rho_n)\subset\mathcal{B}_{\frac{1}{n}}(0;\mathbb{R}^{d-1})\times(-\frac{1}{n},0)$

$$\begin{aligned}&\left|\int_{Q_T}\left((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u)\right)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right|\\ &\leq L_b\left\|\nabla_u f(s,x,u)\right\|_{L^\infty(Q_T;\mathbb{R}^d)}\int_{Q_T}\left(\|y-x\|+\|u-v\|\right)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\\ &\leq L_b\left\|\nabla_u f(s,x,u)\right\|_{L^\infty(Q_T;\mathbb{R}^d)}\left(\int_{\mathcal{D}}\|y-x\|\,\rho_n(y-x)\,dx + \int_{\mathbb{R}^d}\|v-u\|\,g_m(v-u)\,du\right)\\ &\leq L_b\left\|\nabla_u f(s,x,u)\right\|_{L^\infty(Q_T;\mathbb{R}^d)}\left(\frac{1}{n}+\frac{C_{g_1}}{m}\right)\end{aligned} \tag{5.21}$$

where C_{g_1} depends only $\int_{\mathbb{R}^d}\|u\|\,g_1(u)\,du$. Therefore we have that:

$$\left\|R^{\text{Sp}}_{k,n,m}[f](\tau,y,v)\right\|_{L^\infty(Q_T)} \leq C_{\nabla_u f,b,\rho_1,g_1}\left(\frac{n}{m^2}+\frac{1}{m}+\frac{1}{n}\right) \tag{5.22}$$

***ii)* Bound on the derivative of the error in y.**

We consider the first term:

$$\left\|\nabla_y\left(f*\left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot(vg_m))\right)\right)(\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)}=\frac{1}{m^2}\left\|\nabla_y\left(f*\left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot\nabla_v g_m)\right)\right)(\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)}$$
$$=\frac{1}{m^2}\left\|\nabla_v f*\left(\widetilde{\beta}_k\mathrm{Hess}_{y,y}(\rho_n)g_m\right)(\tau,y,v)\right\| L^\infty(Q_T;\mathbb{R}^d)\le C_{\nabla_y^2\rho_1}\|\nabla_u f(s,x,u)\|_{L^\infty(Q_T;\mathbb{R}^d)}\frac{n^2}{m^2} \tag{5.23}$$

where $C_{\nabla_y^2\rho_1}$ depends only on the Hessian of ρ_1.

The second term is bounded using similar arguments as before concerning the fact that b is with bounded derivatives

$$\begin{aligned}&\left\|\nabla_y\int_{Q_T}\left((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u)\right)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right\|_{L^\infty(Q_T;\mathbb{R}^d)}\\&\le\left\|\nabla_y b(y,v)\cdot\int_{Q_T}\nabla_u f(s,x,u)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right\|_{L^\infty(Q_T;\mathbb{R}^d)}\\&\quad+\left\|\int_{Q_T}\left((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u)\right)\widetilde{\beta}_k(\tau-s)\nabla_y\rho_n(y-x)g_m(v-u)\,dsdxdu\right\|_{L^\infty(Q_T;\mathbb{R}^d)}\\&\le\|\nabla_u f\|_{L^\infty(Q_T;\mathbb{R}^d)}\left(\|\nabla_y b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}+C_{\nabla_y\rho_1}+C_{g_1}\frac{n}{m}\right)\end{aligned} \tag{5.24}$$

Therefore we have that:

$$\left\|\partial_y R^{\mathrm{Sp}}_{k,n,m}[f](\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)}\le C_{\nabla_u f,b,\rho_1,g_1}\left(1+\frac{n}{m}+\frac{n^2}{m^2}\right) \tag{5.25}$$

Through a similar procedure, taking the $L^2(Q_T;er^d)-$norm instead of the $L^\infty(Q_T;\mathbb{R}^d)-$norm on $\nabla_u f$, we obtain the desired result.

***iii)* Bound on the derivative of the error in v.**

We consider the first term:

$$\left\|\nabla_v\left(f*\left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot(vg_m))\right)\right)(\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)}=\frac{1}{m^2}\left\|\nabla_v f*\left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot\nabla_v g_m)\right)(\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)}$$
$$\le C_{\nabla_y\rho_1}C_{\nabla_v g_1}\|\nabla_u f(s,x,u)\|_{L^\infty(Q_T;\mathbb{R}^d)}\frac{n}{m} \tag{5.26}$$

where $C_{\nabla_y\rho_1}$ depends only on the gradient of ρ_1 and $C_{\nabla_v g_1}$ on the gradient of g_1.

Following similar calculations in determining the previous bound for the derivative in y, we have that

$$\begin{aligned}&\left\|\nabla_v\int_{Q_T}\left((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u)\right)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right\|\\&\le\|\nabla_u f\|_{L^\infty(Q_T;\mathbb{R}^d)}\left(\|\nabla_v b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}+C_{\rho_1,\nabla_v g_1}\frac{m}{n}+C_{\nabla_v g_1}\right)\end{aligned} \tag{5.27}$$

where $C_{\rho_1,\nabla_v g_1}$ depends on ρ_1 and on the derivative of g_1. Thus we conclude that

$$\left\|\partial_v R^{\mathrm{Sp}}_{k,n,m}[f](\tau,y,v)\right\|\le C_{\nabla_u f,b,\rho_1,g_1}\left(1+\frac{n}{m}+\frac{m}{n}\right) \tag{5.28}$$

Through a similar procedure, taking the $L^2(Q_T;\mathbb{R}^d)-$norm instead of the $L^\infty(Q_T;\mathbb{R}^d)-$norm on $\nabla_u f$, we obtain the desired second bound result.

***iv)* Limit of the error.** We have for any $(\tau,y,v)\in Q_T$:

$$\begin{aligned}&\left|f(T,y,v)-\int_0^T R^{\mathrm{Tm}}_{k,n,m}[f](\tau,y,v)\,d\tau\right|=\left|f(T,y,v)-\int_0^T\widetilde{\beta}(\tau-T)f_{n,m}(T,y,v)\,d\tau dydv\right|\\&=|f(T,y,v)-f_{n,m}(T,y,v)\,d\tau dydv|\xrightarrow[(n,m)\to\infty]{}0\end{aligned} \tag{5.29}$$

v) **Bounds of the derivative of the error.**

$$\int_0^T \left\|\nabla_v R^{\mathrm{Tm}}_{k,n,m}[f](\tau,y,v)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} = \int_0^T \widetilde{\beta}(\tau-T)\left\|\nabla_v f_{n,m}(T,\cdot,\cdot)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \leq \left\|\nabla_v f(T,\cdot,\cdot)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \tag{5.30}$$

and similarly

$$\int_0^T \left\|\nabla_y R^{\mathrm{Tm}}_{k,n,m}[f](\tau,y,v)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} = \int_0^T \widetilde{\beta}(\tau-T)\left\|\nabla_y f_{n,m}(T,\cdot,\cdot)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \leq \left\|\nabla_y f(T,\cdot,\cdot)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \tag{5.31}$$

□

Lemma 5.4. *Assume (H$_{PDE}$). The weak solution f in $L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ to equation (6.11) verifies*

i) $\mathrm{Hess}_{x,u}(f)\in L^2(Q_T;\mathbb{R}^{2d})$,

ii) $\mathrm{Hess}_{u,u}(f)\in L^2(Q_T;\mathbb{R}^{2d})$.

Proof. The proof for these results is based on the equality on $f_{k,n,m}$ from Lemma 5.2. By using an energy equality approach, we obtain a uniform bound in (k,n,m) for $\mathrm{Hess}_{x,u}(f_{k,n,m})$ and we utilise a result from Berzis to conclude.

i) Hessian in x,u

Since $f_{k,n,m}$ is a smooth function on Q_T, we differentiate equality (5.17) with respect to coordinate y_i where y_i is the $i-$th coordinate, to obtain:

$$\begin{aligned}
&-\partial_\tau\partial_{y_i}f_{k,n,m}(\tau,y,v)+(v\cdot\nabla_y\partial_{y_i}f_{k,n,m})(\tau,y,v)+\left(\partial_{y_i}b(y,v)\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\\
&\quad+\left(b(y,v)\cdot\nabla_v\partial_{y_i}f_{k,n,m})\right)(\tau,y,v)+\frac{\sigma^2}{2}\triangle_v\partial_{y_i}f_{k,n,m}(\tau,y,v)=\partial_{y_i}R_{k,n,m}[f](\tau,y,v).
\end{aligned} \tag{5.32}$$

We now multiply this equality by $\partial_{y_i}f_{k,n,m}$ and integrate on Q_T thus obtaining:

$$\begin{aligned}
&-\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\partial_\tau\partial_{y_i}f_{k,n,m}(\tau,y,v)\,d\tau dydv+\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)(v\cdot\nabla_y\partial_{y_i}f_{k,n,m})(\tau,y,v)\,d\tau dydv\\
&+\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\left(\partial_{y_i}b(y,v)\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\,d\tau dydv\\
&+\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\left(b(y,v)\cdot\nabla_v\partial_{y_i}f_{k,n,m})\right)(\tau,y,v)\,d\tau dydv\\
&+\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\frac{\sigma^2}{2}\triangle_v\partial_{y_i}f_{k,n,m}(\tau,y,v)\,d\tau dydv\\
&=\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\partial_{y_i}R_{k,n,m}[f](\tau,y,v)\,d\tau dydv
\end{aligned} \tag{5.33}$$

We now consider each of term of the equation (5.33), starting with the time derivative term

$$\begin{aligned}
&\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\partial_\tau\partial_{y_i}f_{k,n,m}(\tau,y,v)\,d\tau dydv=\frac{1}{2}\int_{Q_T}\partial_\tau\left(\partial_{y_i}f_{k,n,m}\right)^2(\tau,y,v)\,d\tau dydv\\
&=\frac{1}{2}\int_{\mathcal{D}\times\mathbb{R}^d}\left(\partial_{y_i}f_{k,n,m}\right)^2(T,y,v)\,d\tau dydv-\frac{1}{2}\int_{\mathcal{D}\times\mathbb{R}^d}\left(\partial_{y_i}f_{k,n,m}\right)^2(0,y,v)\,d\tau dydv\,.
\end{aligned} \tag{5.34}$$

The second term of equation (5.33) is such that

$$\begin{aligned}
\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)(v\cdot\nabla_y\partial_{y_i}f_{k,n,m})(\tau,y,v)d\tau dydv&=\frac{1}{2}\int_{Q_T}\left(v\cdot\nabla_y\left(\partial_{y_i}f_{k,n,m}\right)^2\right)(\tau,y,v)d\tau dydv\\
&=\frac{1}{2}\int_{\Sigma_T}\left(v\cdot n_{\mathcal{D}(x)}\right)\left\|\partial_{y_i}f_{k,n,m}\right\|^2
\end{aligned}$$

The third term is left as is while the forth term of (5.33) is modified as

$$\begin{aligned}
&\int_{Q_T} \partial_{y_i} f_{k,n,m}(\tau,y,v)\left(b(y,v)\cdot\nabla_v \partial_{y_i} f_{k,n,m})\right)(\tau,y,v)\, d\tau dy dv \\
&= \frac{1}{2}\int_{Q_T}\left(b(y,v)\cdot\nabla_v\left(\partial_{y_i} f_{k,n,m}\right)^2\right)(\tau,y,v) d\tau dy dv = -\frac{1}{2}\int_{Q_T}\left(\nabla_v\cdot b\right)\left(\partial_{y_i} f_{k,n,m}\right)^2(\tau,y,v) d\tau dy dv
\end{aligned}$$

while for the Laplacian term in (5.33) we have that

$$\int_{Q_T} \partial_{y_i} f_{k,n,m}(\tau,y,v)(\nabla_v\cdot\nabla_v)\partial_{y_i} f_{k,n,m}(\tau,y,v)\, d\tau dy dv = -\int_{Q_T}\left\|\nabla_v \partial_{y_i} f_{k,n,m}\right\|^2(\tau,y,v)\, d\tau dy dv\,.$$

Finally, we rewrite (5.33) as

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathcal{D}\times\mathbb{R}^d}\left(\partial_{y_i} f_{k,n,m}\right)^2(T,y,v)\, d\tau dy dv + \frac{1}{2}\int_{\mathcal{D}\times\mathbb{R}^d}\left(\partial_{y_i} f_{k,n,m}\right)^2(0,y,v)\, d\tau dy dv \\
&+\frac{1}{2}\int_{\Sigma_T}\left(v\cdot n_{\mathcal{D}(x)}\right)\left\|\partial_{y_i} f_{k,n,m}\right\|^2 + \int_{Q_T}\partial_{y_i} f_{k,n,m}(\tau,y,v)\left(\partial_{y_i} b(y,v)\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\, d\tau dy dv \\
&-\frac{1}{2}\int_{Q_T}\left(\nabla_v\cdot b\right)\left\|\nabla_y f_{k,n,m}\right\|^2(\tau,y,v) d\tau dy dv - \frac{\sigma^2}{2}\int_{Q_T}\left\|\nabla_v\partial_{y_i} f_{k,n,m}\right\|^2(\tau,y,v)\, d\tau dy dv \\
&= \int_{Q_T}\partial_{y_i} f_{k,n,m}(\tau,y,v)\partial_{y_i} R_{k,n,m}[f](\tau,y,v)\, d\tau dy dv\,.
\end{aligned} \tag{5.35}$$

Summing this equation from $i=1$ to $i=d$ and recalling that

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\left(\partial_{v_j}\partial_{y_i} f_{k,n,m}\right)^2 = \|\mathrm{Hess}_{y,v} f_{k,n,m}\|^2,$$

we obtain that:

$$\begin{aligned}
&-\frac{1}{2}\left\|\nabla_y f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(T) + \frac{1}{2}\left\|\nabla_y f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(0) + \frac{1}{2}\int_{\Sigma_T}\left(v\cdot n_{\mathcal{D}(x)}\right)\left\|\nabla_y f_{k,n,m}\right\|^2 \\
&+\int_{Q_T}\left((\nabla_y f_{k,n,m}(\tau,y,v)\cdot \mathrm{Jac}_y(b)(y,v))\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\, d\tau dy dv \\
&-\frac{1}{2}\int_{Q_T}\left(\nabla_v\cdot b\right)\left\|\nabla_y f_{k,n,m}\right\|^2(\tau,y,v) d\tau dy dv - \frac{\sigma^2}{2}\left\|\mathrm{Hess}_{y,v}(f_{k,n,m})\right\|^2_{L^2(Q_T;\mathbb{R}^{2d})} \\
&= \int_{Q_T}\left(\nabla_y f_{k,n,m}\cdot\nabla_y R_{k,n,m}[f]\right)(\tau,y,v)\, d\tau dy dv\,.
\end{aligned} \tag{5.36}$$

which we reorganise as

$$\begin{aligned}
\frac{\sigma^2}{2}\left\|\mathrm{Hess}_{y,v}(f_{k,n,m})\right\|^2_{L^2(Q_T;\mathbb{R}^{2d})} &= -\frac{1}{2}\left\|\nabla_y f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(T) + \frac{1}{2}\left\|\nabla_y f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(0) \\
&+\frac{1}{2}\int_{\Sigma_T}\left(v\cdot n_{\mathcal{D}(x)}\right)\left\|\nabla_y f_{k,n,m}\right\|^2 + \int_{Q_T}\left((\nabla_y f_{k,n,m}(\tau,y,v)\cdot \mathrm{Jac}_y(b)(y,v))\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\, d\tau dy dv \\
&-\frac{1}{2}\int_{Q_T}\left(\nabla_v\cdot b\right)\left\|\nabla_y f_{k,n,m}\right\|^2(\tau,y,v) d\tau dy dv - \int_{Q_T}\left(\nabla_y f_{k,n,m}\cdot\nabla_y R_{k,n,m}[f]\right)(\tau,y,v)\, d\tau dy dv\,.
\end{aligned} \tag{5.37}$$

We have the following inequality:

$$\begin{aligned}
\frac{\sigma^2}{2}\left\|\mathrm{Hess}_{y,v}(f_{k,n,m})\right\|^2_{L^2(Q_T;\mathbb{R}^{2d})} &\le \frac{1}{2}\left\|\nabla_y f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(0) + \frac{1}{2}\int_{\Sigma_T}\left(v\cdot n_{\mathcal{D}(x)}\right)\left\|\nabla_y f_{k,n,m}\right\|^2 \\
&+\int_{Q_T}\left((\nabla_y f_{k,n,m}(\tau,y,v)\cdot \mathrm{Jac}_y(b)(y,v))\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\, d\tau dy dv \\
&-\frac{1}{2}\int_{Q_T}\left(\nabla_v\cdot b\right)\left\|\nabla_y f_{k,n,m}\right\|^2(\tau,y,v) d\tau dy dv - \int_{Q_T}\left(\nabla_y f_{k,n,m}\cdot\nabla_y R_{k,n,m}[f]\right)(\tau,y,v)\, d\tau dy dv\,.
\end{aligned} \tag{5.38}$$

and we bound each of the terms in the r.h.s. of (5.38), uniformly in (k,l,m) using the regularity of the function f obtained from Lemma 4.6

$$\|\nabla_y f_{k,n,m}\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(0) \le \sup_{t\in[0,T]} \|\nabla_y f_{n,m}\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(t) \le \sup_{t\in[0,T]} \|\nabla_y f\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(t)\,.$$

By (H_{PDE}), the derivatives of the function b are bounded, then by Cauchy-Schwartz:

$$\begin{aligned}
&\left|\int_{Q_T} \left(\left(\nabla_y f_{k,n,m}(\tau,y,v)\cdot \mathrm{Jac}_y(b)(y,v)\right)\cdot \nabla_v f_{k,n,m}\right)(\tau,y,v)\,d\tau dy dv\right| \\
&\le \|\mathrm{Jac}_y(b)\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^{2d})} \|\nabla_y f_{k,n,m})\|_{L^2(Q_T;\mathbb{R}^d)} \|\nabla_v f_{k,n,m})\|_{L^2(Q_T;\mathbb{R}^d)} \\
&\le \|\mathrm{Jac}_y(b)\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^{2d})} \|\nabla_y f\|_{L^2(Q_T;\mathbb{R}^d)} \|\nabla_v f\|_{L^2(Q_T;\mathbb{R}^d)}
\end{aligned}$$

while

$$\left|\int_{Q_T} (\nabla_v\cdot b)\,\|\nabla_y f_{k,n,m}\|^2(\tau,y,v)d\tau dy dv\right| \le \|\nabla_y\cdot b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)} \|\nabla_y f\|^2_{L^2(Q_T;\mathbb{R}^d)}\,.$$

We now consider Lemma 5.3 to control the errors as:

$$\begin{aligned}
&\left|\int_{Q_T} \left(\nabla_y f_{k,n,m}\cdot \nabla_y R_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\right| \\
&\le \left|\int_{Q_T} \left(\nabla_y f_{k,n,m}\cdot \nabla_y R^{\mathrm{Sp}}_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\right| + \left|\int_{Q_T} \left(\nabla_y f_{k,n,m}\cdot \nabla_y R^{\mathrm{Tm}}_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\right| \\
&\le \|\nabla_y f\|_{L^2(Q_T;\mathbb{R}^d)} \left\|\nabla_y R^{\mathrm{Sp}}_{k,n,m}[f]\right\|_{L^2(Q_T;\mathbb{R}^d)} + \sup_{t\in[0,T]} \|\nabla_y f_{k,n,m}(t,\cdot,\cdot)\|_{L^2(\mathcal{D}\times\mathbb{R}^2;\mathbb{R}^d)} \int_0^T \left\|\nabla_y R^{\mathrm{Tm}}_{k,n,m}[f]\right\|_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)} \\
&\le C_{\nabla_y f}\|\nabla_y f\|_{L^2(Q_T;\mathbb{R}^d)} + C_{\nabla_y f}\sup_{t\in[0,T]} \|\nabla_y f\|_{L^2(\mathcal{D}\times\mathbb{R}^D;\mathbb{R}^d)}\,.
\end{aligned} \tag{5.39}$$

By combining these various bounds and going back to inequality (5.38) we obtain that

$$\frac{\sigma^2}{2}\|\mathrm{Hess}_{y,v}(f_{k,n,m})\|^2_{L^2(Q_T;\mathbb{R}^{2d})} \le C_{\nabla_y f,\nabla_v f,b,\nabla_y b} + \frac{1}{2}\|\nabla_y f_{k,n,m}\|^2_{L^2(\Sigma_T)} \tag{5.40}$$

where $C_{\nabla_y f,\nabla_v f,b,\nabla_y b}$ does not depend on (k,n,m). By Lemma 4.6, we have that $\|\nabla_y f\|^2_{L^2(\Sigma_T)}$ is finite, therefore $\mathrm{Hess}_{y,v}(f_{k,n,m})$ is bounded in $L^2(Q_T;\mathbb{R}^{2d})$. Since $\nabla_y f_{k,l,m}$ and $\nabla_v f_{k,l,m}$ converge in $L^2(Q_T;\mathbb{R}^d)$, by [9], we obtain that $\mathrm{Hess}_{y,v}(f)\in L^2(Q_T;\mathbb{R}^{2d})$.

ii) Hessian in u,u

We now prove a similar result for the second derivative w.r.t. u. We apply the same calculations: differentiate equality (5.17) with respect to coordinate v_i where v_i is the $i-$th coordinate, multiplying by $\partial_{v_i} f_{k,n,m}$ and integrating over Q_T, we obtain:

$$\begin{aligned}
&-\int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\partial_\tau\partial_{v_i} f_{k,n,m}(\tau,y,v)\,d\tau dy dv + \int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\partial_{y_i} f_{k,n,m}(\tau,y,v)\,d\tau dy dv \\
&\quad+\int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)(v\cdot\nabla_y\partial_{v_i} f_{k,n,m})(\tau,y,v)\,d\tau dy dv \\
&\quad+\int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\left(\partial_{v_i} b(y,v)\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\,d\tau dy dv \\
&\quad+\int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\left(b(y,v)\cdot\nabla_v\partial_{v_i} f_{k,n,m})\right)(\tau,y,v)\,d\tau dy dv \\
&\quad+\int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\frac{\sigma^2}{2}\triangle_v\partial_{v_i} f_{k,n,m}(\tau,y,v)\,d\tau dy dv \\
&=\int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\partial_{v_i} R_{k,n,m}[f](\tau,y,v)\,d\tau dy dv\,.
\end{aligned} \tag{5.41}$$

Now we sum for $i = 1$ to $i = d$ and integrate by parts as in the previous section to obtain that

$$
\begin{aligned}
&-\frac{1}{2}\left\|\nabla_v f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(T) + \frac{1}{2}\left\|\nabla_v f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(0) + \int_{Q_T}\left(\nabla_y f_{k,n,m}\cdot\nabla_v f_{k,n,m}\right)(\tau,y,v)\,d\tau dy dv \\
&+\frac{1}{2}\int_{\Sigma_T}(v\cdot n_{\mathcal{D}(x)})\left\|\nabla_v f_{k,n,m}\right\|^2 + \int_{Q_T}\left(\left(\nabla_v f_{k,n,m}(\tau,y,v)\cdot \mathrm{Jac}_v(b)(y,v)\right)\cdot\nabla_v f_{k,n,m}\right)(\tau,y,v)\,d\tau dy dv \\
&-\frac{1}{2}\int_{Q_T}(\nabla_v\cdot b)\left\|\nabla_v f_{k,n,m}\right\|^2(\tau,y,v)d\tau dy dv - \frac{\sigma^2}{2}\left\|\mathrm{Hess}_{v,v}(f_{k,n,m})\right\|^2_{L^2(Q_T;\mathbb{R}^{2d})} \\
&= \int_{Q_T}\left(\nabla_v f_{k,n,m}\cdot\nabla_v R_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\,.
\end{aligned}
\tag{5.42}
$$

By using the analogous arguments as previously, we obtain that $\mathrm{Hess}_{v,v}(f_{k,n,m})$ is bounded in $L^2(Q_T;\mathbb{R}^{2d})$ and since $\nabla_v f_{k,n,m}$ converges in $L^2(Q_T;\mathbb{R}^d)$, we obtain by [9], that $\mathrm{Hess}_{v,v}(f)\in L^2(Q_T;\mathbb{R}^{2d})$.

□

Corollary 5.5. *Assume (H_{PDE}). The weak solution F to equation (1.9) verifies that* $\mathrm{Hess}_{x,u}(F), \mathrm{Hess}_{u,u}(F)\in L^2(Q_T;\mathbb{R}^{2d})$.

Proof. By the previous lemma, we have that $\mathrm{Hess}_{x,u}(f), \mathrm{Hess}_{u,u}(f)\in L^2(Q_T,\mathbb{R}^{2d})$. And since for any $(t,x,u)\in Q_T$, $f(t,x,u) = F(T-t,x,u)$, we have that:

$$
\begin{aligned}
&\int_0^T\int_{\mathcal{D}\times\mathbb{R}^d}\left\|\mathrm{Hess}_{x,u}(f)\right\|^2_F(t,x,u)\,dtdxdu = \int_0^T\int_{\mathcal{D}\times\mathbb{R}^d}\left\|\mathrm{Hess}_{x,u}(F)\right\|^2_F(T-t,x,u)\,dtdxdu \\
&= -\int_T^0\int_{\mathcal{D}\times\mathbb{R}^d}\left\|\mathrm{Hess}_{x,u}(F)\right\|^2_F(s,x,u)\,dsdxdu = \left\|\mathrm{Hess}_{x,u}(F)\right\|^2_{L^2(Q_T,\mathbb{R}^{2d})} < +\infty
\end{aligned}
$$

by performing the change of variable $s = T-t$. The same argument gives that $\mathrm{Hess}_{u,u}(F)\in L^2(Q_T;\mathbb{R}^{2d})$.

□

6 On the semigroup of the confined Langevin process

In this section we present several results that pertain to the existence and regularity of the weak solution of the PDE (1.9). Without any loss of generality, we consider the time forward formulation of this PDE, written is its variational formulation in (5.1). This section is an extract from [6] with minor modifications to include a bounded Lipschitz drift b in the PDE problem (6.5). We first assume that b is a smooth function and then we come back to our hypothesis (H_{PDE}). The proofs are transferred to the AppendixC.

We investigate some estimates related to the semigroup associated to the solution of the SDE (1.1); namely, for a test function $\psi\in\mathcal{C}_c^\infty(\mathcal{D}\times\mathbb{R}^d)$, for all $(x,u)\in(\mathcal{D}\times\mathbb{R}^d)\cup(\Sigma\setminus\Sigma^0)$, we define

$$
\Gamma^\psi(t,x,u) := \mathbb{E}_{\mathbb{P}}\left[\psi(X_t^{x,u},U_t^{x,u})\right], \tag{6.1}
$$

where $((X_t^{x,u},U_t^{x,u});\, t\in[0,T])$ is the solution of (1.1) starting from $(0,x,u)$ and $((X_t^{s,x,u},U_t^{s,x,u});\, t\in[0,T])$ is the solution of (1.1) starting from (s,x,u).

Pathwise uniqueness of the confined Langevin process implies that for all $0\le s\le t\le T$,

$$
\Gamma^\psi(t-s,x,u) = \mathbb{E}_{\mathbb{P}}\left[\psi(X_t^{s,x,u},U_t^{s,x,u})\right], \tag{6.2}
$$

so that the estimates hereafter can be extended to the semigroup transitions of the process. We can see that $\Gamma^\psi(T-s,x,u) = F(s,x,u)$.

We consider also the semigroup related to the stopped process:

$$
\Gamma_n^\psi(t,x,u) = \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_{t\wedge\tau_n^{x,u}},U^{x,u}_{t\wedge\tau_n^{x,u}})\right], \tag{6.3}
$$

where $\{\tau_n^{x,u}; n\in\mathbb{N}\}$ is the sequence of hitting times defined as

$$
\tau_n = \inf\{\tau_{n-1} < t\le T\,;\ X_t\in\partial\mathcal{D}\},\ \text{for } n\ge 1,\quad \tau_0 = 0,
$$

and $\Gamma^{\psi}_0(t,x,u)=\psi(x,u)$.

When b is a smooth function, the estimates on $\{\Gamma^{\psi}_n;\, n\geq 1\}$ and Γ^{ψ} rely on the following PDE result, the proof of which is postponed in the next Subsection C.1. Let $((x^{y,v}_t,u^{y,v}_t);\, t\in[0,T])$ be the free Langevin process that verifies

$$\begin{cases} x^{y,v}_t = y + \int_0^t u^{y,v}_s ds, \\ u^{y,v}_t = v + \int_0^t \widetilde{b}(x^{y,v}_s,u^{y,v}_s)\, ds + \sigma W_t, \end{cases} \tag{6.4}$$

where $\tilde{b}$ is defined in (4.2).

Theorem 6.1. *Assume (H_{PDE}). Assume also that b is a $\mathcal{C}^{\infty}_b(\mathbb{R}^d\times\mathbb{R}^d;\mathbb{R}^d)$ function. Given two functions $f_0\in L^2(\mathcal{D}\times\mathbb{R}^d)\cap\mathcal{C}_b(\mathcal{D}\times\mathbb{R}^d)$ and $q\in L^2(\Sigma^+_T)\cap\mathcal{C}_b(\Sigma^+_T)$, there exists a unique function $f\in\mathcal{C}^{1,1,2}_b(Q_T)\cap\mathcal{C}((0,T]\times(\overline{\mathcal{D}}\times\mathbb{R}^d\setminus\Sigma^0))\cap L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ which is a solution to*

$$\begin{cases} \partial_t f(t,x,u) - (u\cdot\nabla_x f(t,x,u)) - (b(x,u)\cdot\nabla_u f(t,x,u)) - \dfrac{\sigma^2}{2}\triangle_u f(t,x,u)=0, \ \text{for all } (t,x,u)\in Q_T, \\ f(0,x,u)=f_0(x,u), \ \text{for all } (x,u)\in\mathcal{D}\times\mathbb{R}^d, \\ f(t,x,u)=q(t,x,u), \ \text{for all } (t,x,u)\in\Sigma^+_T. \end{cases} \tag{6.5}$$

In addition, for $(x^{x,u}_t,u^{x,u}_t;\, t\in[0,T])$ solution to (6.4) starting from $(x,u)\in\mathcal{D}\times\mathbb{R}^d$ at $t=0$ and $\beta^{x,u}:=\inf\{t>0\,;\, x^{x,u}_t\in\partial\mathcal{D}\}$, we have

$$f(t,x,u)=\mathbb{E}_{\mathbb{P}}\left[f_0(x^{x,u}_t,u^{x,u}_t)\mathbb{1}_{\{t\,\leq\,\beta^{x,u}\}}\right]+\mathbb{E}_{\mathbb{P}}\left[q(t-\beta^{x,u},x^{x,u}_{\beta^{x,u}},u^{x,u}_{\beta^{x,u}})\mathbb{1}_{\{t\,>\,\beta^{x,u}\}}\right]. \tag{6.6}$$

Furthermore, for all $t\in(0,T)$, f satisfies the inequality:

$$\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}+\|f\|^2_{L^2(\Sigma^-_t)}+\sigma^2\|\nabla_u f\|^2_{L^2(Q_t)}\leq C_{T,\sigma,\|b\|_{\infty,Lip}}\left(\|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}+\|q\|^2_{L^2(\Sigma^+_t)}\right) \tag{6.7}$$

where $C_{T,\sigma,\|b\|_{\infty,Lip}}$ is a constant that only depends on T, σ, and on the Lipschitz constant in u, uniform in x of b, $\|b\|_{\infty,Lip}$.

The proof of this theorem is split in several lemmas and propositions in Appendix C.1. In Lemma C.1, we prove the L^p regularity of the solution together with the energy inequality. It is based on the Lions and Magenes' existence theorem stated in A.3 and on Carrillo's trace existence and Green formula in A.4. For the inner regularity of the solution, Bouchut's Theorem A.5 is used to obtain fractional L^p regularity, while bootstrapping techniques are used to increase this regularity to obtain Sobolev estimates to obtain embeddings into continuous spaces in proposition C.2. Continuity up to the boundary Σ^+_T is proven using local barrier functions in proposition C.4 while continuity up to the border Σ^-_T is proven in proposition C.5 using the Feynman-Kac interpretation (6.6).

Considering the solution f in $\mathcal{C}([0,T];L^2(\mathcal{D}\times\mathbb{R}^d))\cap\mathcal{H}(Q_T)$ of (C.1), given by Lemma C.1, we show its interior regularity and its continuity up to and along $\Sigma_T\setminus\Sigma^0_T$.

From Theorem 6.1, we deduce the following result for $\{\Gamma^{\psi}_n, n\geq 1\}$:

Corollary 6.2. *Assume (H_{PDE}). Assume also that b is a $\mathcal{C}^{\infty}_b(\mathbb{R}^d\times\mathbb{R}^d)$ function. Then, for all $\psi\in\mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d)$, set $\Gamma^{\psi}_0=\psi$ and for all $n\in\mathbb{N}^*$, Γ^{ψ}_n is a function in $\mathcal{C}^{1,1,2}_b(Q_T)\cap\mathcal{C}(\overline{Q_T}\setminus\Sigma^0)$ and satisfies the PDE*

$$\begin{cases} \partial_t\Gamma^{\psi}_n(t,x,u) - (u\cdot\nabla_x\Gamma^{\psi}_n(t,x,u)) - (b(x,u)\cdot\nabla_u\Gamma^{\psi}_n(t,x,u)) - \dfrac{\sigma^2}{2}\triangle_u\Gamma^{\psi}_n(t,x,u)=0, \ \text{for all } (t,x,u)\in Q_T, \\ \Gamma^{\psi}_n(0,x,u)=\psi(x,u), \ \text{for all } (x,u)\in\mathcal{D}\times\mathbb{R}^d, \\ \Gamma^{\psi}_n(t,x,u)=\Gamma^{\psi}_{n-1}(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \ \text{for all } (t,x,u)\in\Sigma^+_T. \end{cases} \tag{6.8}$$

In addition, the set $\{\Gamma^{\psi}_n, n\geq 1\}$ belongs to $L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ and satisfies the inequality

$$\|\Gamma^{\psi}_n(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}+\sigma^2\|\nabla_u\Gamma^{\psi}_n\|^2_{L^2(Q_t)}+\|\Gamma^{\psi}_n\|^2_{L^2(\Sigma^-_t)}\leq C_{T,\sigma,\|b\|_{\infty,Lip}}\left(\|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}+\|\Gamma^{\psi}_{n-1}\|^2_{L^2(\Sigma^-_t)}\right) \tag{6.9}$$

where $C_{T,\sigma,\|b\|_{\infty,Lip}}$ is a constant that only depends on T, b and σ.

The proof of this corollary is based on the Theorem 6.1. The unique solution to equation (6.5) with initial condition ψ and boundary condition $\Gamma^{\psi}_{n-1}(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))$ when written under its probabilistic interpretation (6.6) is actually equal to Γ^{ψ}_{n} defined in (6.3).

Next, by showing the convergence of the Γ^{ψ}_{n} to Γ^{ψ}, we have

Corollary 6.3. *Assume (H_{PDE}). Assume also that b is a $\mathcal{C}^{\infty}_b(\mathbb{R}^d\times\mathbb{R}^d)$ function. For all $\psi\in\mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d)$, Γ^{ψ} is a function that belongs to $L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ and satisfies the inequality:*

$$\|\Gamma^{\psi}(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}+\sigma^2\|\nabla_u\Gamma^{\psi}\|^2_{L^2(Q_t)}\leq C_{T,\sigma,\|b\|_{\infty,Lip}}\|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)},\ \forall\, t\in(0,T) \tag{6.10}$$

where $C_{T,\sigma,\|b\|_{\infty,Lip}}$ is a positive constant that depends only on T, $\|\nabla_u\cdot b\|_{\infty}$ and σ. Furthermore, $\Gamma^{\psi}(t)$ is solution in the sense of distributions of

$$\begin{cases}\partial_t\Gamma^{\psi}-(u\cdot\nabla_x\Gamma^{\psi})-(b(x,u)\cdot\nabla_u\Gamma^{\psi})-\dfrac{\sigma^2}{2}\triangle_u\Gamma^{\psi}=0,\ \textit{on } Q_T,\\ \Gamma^{\psi}(0,x,u)=\psi(x,u),\ \textit{on } \mathcal{D}\times\mathbb{R}^d,\\ \Gamma^{\psi}(t,x,u)=\Gamma^{\psi}(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)),\ \textit{on } \Sigma^+_T.\end{cases} \tag{6.11}$$

The proof of this corollary is given in the Appendix C.3.

Finally, the following proposition allows to extend the energy estimate (6.10) to the case of drift b satisfying only (H_{PDE}).

Proposition 6.4. *Assume only (H_{PDE}). Then for all $\psi\in\mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d)$, Γ^{ψ} defined in (6.1) is a function that belongs to $L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ and satisfies the inequality (6.10). Furthermore, $\Gamma^{\psi}(t)$ is solution in the sense of distributions of Equation (6.11).*

Proof. We construct the family $\{b_n,n\in\mathbb{N}\}$ of smooth approximation of b by the following convolution product: for any (x,u) in $\mathcal{D}\times\mathbb{R}^d$,

$$b_n(x,u)=\int_{\mathcal{D}\times\mathbb{R}^d}g_n(u-v)\rho_n(x-y)b(y,v)dydv,$$

where the smoothing kernels g and ρ are as in (3.4) and (3.3), (eventually with the d-product of each kernels to expend the definition to the dimension d). We then define the symetrized extension $\tilde{b}_n$ of b_n on $\mathbb{R}^d\times\mathbb{R}^d$ by

$$\tilde{b}_n\colon(y,v)\in\mathbb{R}^d\times\mathbb{R}^d\mapsto\left(b'_n,\operatorname{sign}(y^{(d)})b^{(d)}_n\right)\left((y',|y^{(d)}|),(v',\operatorname{sign}(y^{(d)})v^{(d)})\right), \tag{6.12}$$

and we consider the family of processes $(X^n_t,U^n_t,t\in[0,T])$ and $(Y^n_t,V^n_t,t\in[0,T])$, solution for each fixed n, to the SDEs (1.1) and (1.2), where we have replaced b and $\tilde{b}$ respectively by b_n and $\tilde{b}_n$.

It is classical to observe that b_n inherits from the Lipschitz property of b, with the same constant $\|b\|_{\text{Lip}}$, preserved by the smoothing convolution uniformly in n. Reproducing the arguments in Remark 1.4, we can also deduce that $\tilde{b}_n$ is uniformly Lipschitz on $\mathbb{R}^d\times\mathbb{R}^d$ with constant $2\|b\|_{\text{Lip}}$, and that $\tilde{b}_n$ converges to $\tilde{b}$ uniformly on $\mathbb{R}^d\times\mathbb{R}^d$.

Then the family of processes $(Y^n_t,V^n_t,t\in[0,T])$ belongs in $L^2(\Omega)$ uniformly in time with

$$\begin{aligned}\sup_{n\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\left[\|Y^n_t\|^2+\|V^n_t\|^2\right]&\leq C(T,\|b\|_{\infty},\|b\|_{\text{Lip}}),\\ \mathbb{E}\left[\|Y^n_t-Y^n_s\|^2\right]&\leq C'(T,\|b\|_{\infty},\|b\|_{\text{Lip}})|t-s|.\end{aligned}$$

From the relative compactness property, renaming again $(Y^n_t,V^n_t,t\in[0,T])$ a converging sub-sequence with limit $(Y^{\infty}_t,V^{\infty}_t,t\in[0,T])$, and from the convergence of $\tilde{b}_n$ to $\tilde{b}$, we check that Y^{∞} satisfies (1.2) with drift $\tilde{b}$. By the uniqueness of the solution of (1.2) and also (1.1), we deduce that, for all $t\in[0,T]$,

$$f_n(t,x,u)=\mathbb{E}_{x,u}[\overline{\psi}(Y^n_t,V^n_t)]\xrightarrow[n\to+\infty]{}\mathbb{E}_{x,u}[\overline{\psi}(Y_t,V_t)]=\mathbb{E}_{x,u}[\psi(X_t,U_t)]=\Gamma^{\psi}(t,x,u)$$

for Γ^{ψ} defined in (6.1), since the discontinuity points of $(x,u)\mapsto\overline{\psi}(x,u)$ are $\mathcal{P}\circ(Y_t,V_t)^{-1}$-negligible.

Now by applying Corollary 6.3 to f_n and taking the limit with n, we deduce immediately that Γ^{ψ} is solution to (6.11) in the distribution sense. In particular by Fatou Lemma, the $(\nabla_u f_n,n\geq 0)$ are converging in $L^2(Q_T)$, as n tends to infinity, defining $\nabla_u\overline{\psi}$ as its $L^2(Q_T)$-limit and the Energy inequality (C.17) is preserved. Using the variational formulation of equation (6.11) in the Appendix C.3, we deduce that Γ^{ψ} is a $\mathcal{H}(Q_T)$-solution of (6.11) with trace functions $\gamma^{\pm}(\Gamma^{\psi})$ in $L^2(\Sigma^{\pm})$. ☐

Appendices

A Some recalls

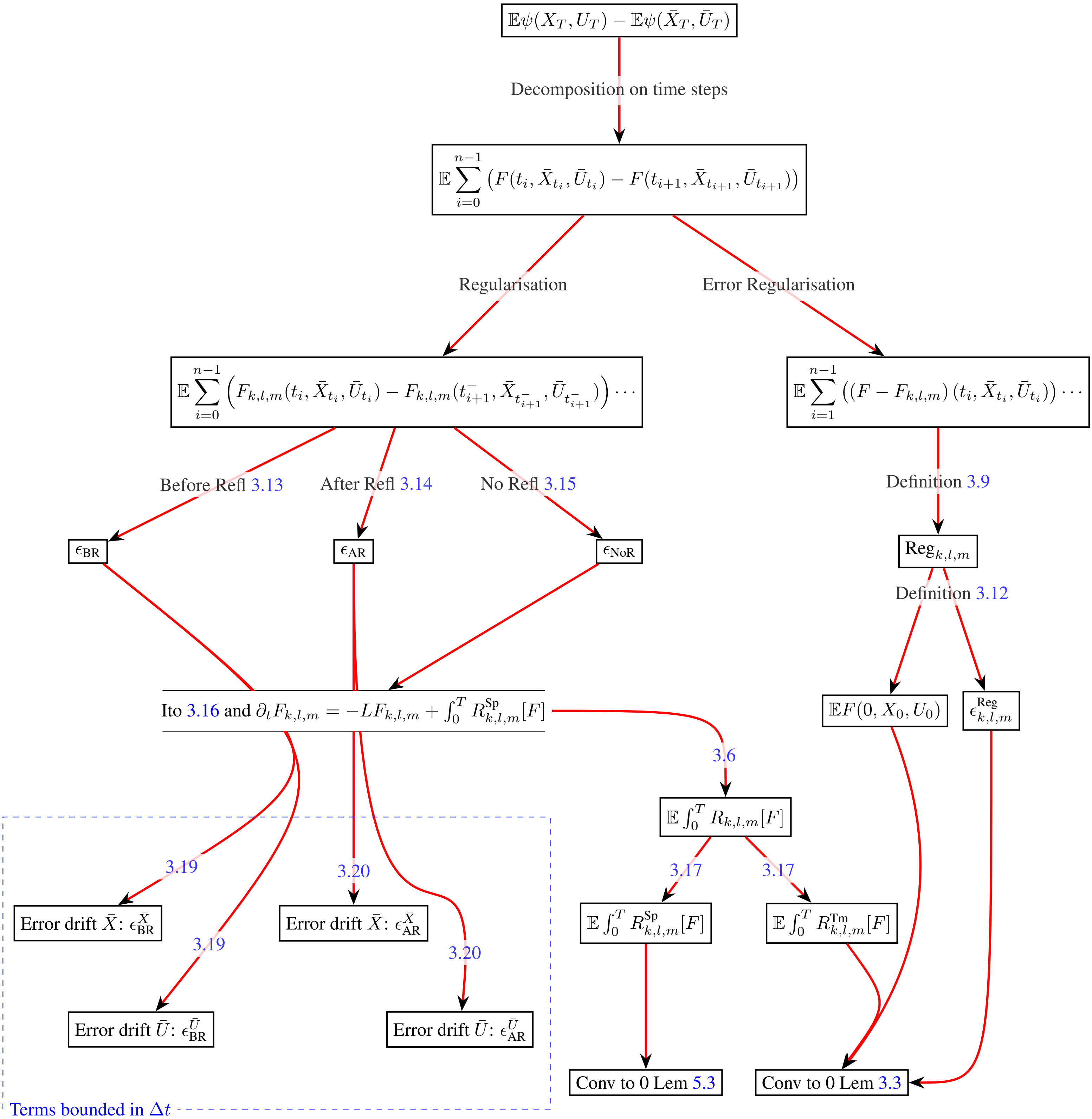

Figure 2: General Schematic of introduced definitions in the main Theorem 1.6

Corollary A.1 (Rana [24]). *If $\phi \in L^p(\mathbb{R}^d)$ for $p \in [1, +\infty)$ then*

$$\lim_{|\delta|\to 0+} \int |\phi(z+\delta) - \phi(z)|^p dz = 0.$$

Theorem A.2 (Tartar [26], Chapter 4). *Let $\mathcal{V}$ be an open subset of $\mathbb{R}^d$ and $\psi \in L^2(\mathcal{V})$ such that $\nabla_v \psi \in L^2(\mathcal{V})$. Then $\nabla_v(\psi)^+, \nabla_v(\psi)^- \in L^2(\mathcal{V})$ with $\partial_{v_i}(\psi)^+ = \partial_{v_i}\psi \mathbb{1}_{\{\psi\geq 0\}}$ and $\partial_{v_i}(\psi)^- = -\partial_{v_i}\psi\mathbb{1}_{\{\psi\leq 0\}}$.*

Theorem A.3 (Lions and Magenes [19]). *Let E be a Hilbert space with the inner product $(\ ,\ )_E$. Let $F \subset E$ equipped with the norm $|\ |_F$ such that the canonical injection of F into E is continuous. Assume that $A : E \times F \to \mathbb{R}$ is a bilinear application satisfying:*

1. *$\forall\, \psi \in F$, the mapping $A(.,\psi) : E \to \mathbb{R}$ is continuous.*
2. *A is coercive on F that is there exists a constant $c > 0$ such that $A(\psi,\psi) \;\geq\; c|\psi|_F^2$, $\forall\, \psi \in F$.*

Then for all linear application $L \ :\ F \to \mathbb{R}$, continuous on $(F, |\ |_F)$, there exists $S \in E$ such that $A(S,\psi) = L(\psi)$, $\forall\, \psi \in F$.

Let $\mathcal{T} = \partial_t - u\nabla_x$ be the transport operator and consider the space:

$$\mathcal{Y}(Q_T) = \{\varphi \in \mathcal{H}(Q_T); -\mathcal{T}(\varphi) \in \mathcal{H}'(Q_T)\}.$$

Theorem A.4 (Carrillo [10]). *For any $T > 0$, we have that:*

1. *Let $\varphi \in \mathcal{Y}(Q_T)$. Then:*
   - *φ has a trace $\gamma^+(\varphi) \in L^2(\Sigma_T^+)$ on Σ_T^+ and $\gamma^-(\varphi) \in L^2(\Sigma_T^-)$ on Σ_T^-.*
   - *$\forall t \in [0,T]$, φ has a trace $\varphi(t,\cdot)$ such that the function $t \mapsto \varphi(t,\cdot)$ belongs to $L^2(\mathcal{D}\times\mathbb{R}^d)$.*
2. *For any functions φ, ψ belonging to $\mathcal{Y}(Q_T)$, we have the following Green formula, for any $t \in [0,T]$:*

$$\begin{aligned}(\mathcal{T}(\varphi),\psi)_{\mathcal{H}'(Q_T),\mathcal{H}(Q_T)} - (\mathcal{T}^*(\psi),\varphi)_{\mathcal{H}'(Q_T),\mathcal{H}(Q_T)} &= \int_{\mathcal{D}\times\mathbb{R}^d} \varphi(T,x,u)\psi(T,x,u)\,dxdu \\ - \int_{\mathcal{D}\times\mathbb{R}^d} \varphi(0,x,u)\psi(0,\cdot,\cdot)\,dxdu &- \int_{\Sigma_T^+} (u\cdot n_{\mathcal{D}})\gamma^+(\varphi)(s,x,u)\gamma^+(\psi)(s,x,u)\,d\lambda_\Sigma(s,x,u) \\ - \int_{\Sigma_T^-} (u\cdot n_{\mathcal{D}})\gamma^-(\varphi)(s,x,u)\gamma^-(\psi)(s,x,u)\,d\lambda_\Sigma(s,x,u)\end{aligned} \tag{A.1}$$

   where $\mathcal{T}^* = -\partial_t + u\cdot\nabla_x$, the adjoint of $\mathcal{T}$.

Theorem A.5 (Bouchut [7]). *Let $h \in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$. Assume that $\phi \in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$, such that $\nabla_u\phi \in (L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))^d$, satisfies (in the sense of distributions)*

$$\partial_t\phi + (u\cdot\nabla_x\phi) - \frac{\sigma^2}{2}\triangle_u\phi = h, \text{ on } \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d. \tag{A.2}$$

Then there exists a positive constant $C(d)$ depending on the dimension such that:

(a) *$\partial_t\phi + (u\cdot\nabla_x\phi)$ and $\triangle_u\phi$ both belong to $L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$ with*

$$\|\partial_t\phi + (u\cdot\nabla_x\phi)\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} + \frac{\sigma^2}{2}\|\triangle_u\phi\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \leq C(d)\|h\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)},$$

(b) *$D_x^{2/3}\phi$ and $|\nabla_u D_x^{1/3}\phi|$ belong to $L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$ with*

$$\|\nabla_u D_x^{1/3}\phi\|^2{}_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} + \|D_x^{2/3}\phi\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \leq C(d)\|h\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}.$$

where for $\alpha \in (0,1)$, D_x^α is the fractional derivative w.r.t. x-variables, defined as the fractional Laplace operator of order α defined as $D_x^\alpha = (-\triangle_x)^{\alpha/2}$.

Lemma A.6. *Let $T > 0$. Consider the mollifying sequence β_k such that* $\mathrm{Supp}(\beta_k) \subset (0, \frac{T}{k})$ *and assume the function $F\colon [0,T] \mapsto \mathbb{R}$ is continuous on $[0,T]$. Then the convolution $F * \beta_k$ converges uniformly towards F on any compact of* $(0,T]$.

Proof. We extend the function F continuously on $\mathbb{R}$ and we denote this continuation as $\widetilde{F}$. By [9], we have that $\widetilde{F} * \beta_k$ converges uniformly towards $\widetilde{F}$.

Let K_ε be a compact of $(0,T]$ such that the distance $d(0, K_\varepsilon) \geq \varepsilon$. On K_ε, $\widetilde{F} * \beta_k$ converges uniformly towards $\widetilde{F}$. For large enough $k \geq k_\varepsilon$, $\mathrm{Supp}(\beta_k) \cap K_\varepsilon = \emptyset$, and by comparing the supports, for any $t \in K_\varepsilon$, $\widetilde{F} * \beta_k(t) = F * \beta_k(t)$. Let $k \geq k_\varepsilon$:

$$\sup_{t \in K_\varepsilon} \left| \widetilde{F} * \beta_k(t) - \widetilde{F}(t) \right| = \sup_{t \in K_\varepsilon} |F * \beta_k(t) - F(t)| \xrightarrow{k \to \infty} 0$$

and we conclude. □

B Complement to Lemma 2.2 about the density of the discretized free Langevin process

Lemma B.1. *The transition density of the discretized version of the free Langevin process*

$$\begin{cases} \bar{Z}_t = x + tu + \sigma \displaystyle\int_0^t W_{\eta(s)}\, ds \\ \bar{V}_t = u + \sigma W_t \end{cases} \tag{B.1}$$

is a Gaussian transition density

$$\bar{p}^L(0; x, u; t; \xi, \zeta) = p_{\mathcal{N}(0, \Sigma_{t,\eta(t),\Delta t})}(\xi - (x + tu), \zeta - u)$$

where $p_{\mathcal{N}(0,\Gamma)}$ denotes the centered Gaussian density with covariance Γ, and

$$\Sigma_{t,\eta(t),\Delta t} = \sigma^2 \begin{bmatrix} t\eta(t)(t - \eta(t) - \Delta t) + \frac{\eta(t)(\eta(t)+\Delta t)(2\eta(t)+\Delta t)}{6} & t\eta(t) - \frac{\eta(t)(\eta(t)+\Delta t)}{2} \\ t\eta(t) - \frac{\eta(t)(\eta(t)+\Delta t)}{2} & t \end{bmatrix}$$

is degenerate in its first coordinate when $t < \Delta t$.

Proof. We have that $\mathbb{E}\bar{Z}_t = x + tu$ and $\mathbb{E}\bar{V}_t = u$. Also $\mathbb{V}\mathrm{ar}\left[\bar{V}_t\right] = \sigma^2 t$ and

$$\int_0^t W_{\eta(s)}\, ds = \sum_{i=0}^{\lfloor \frac{t}{\Delta t} \rfloor - 1} W_{t_i}(t_{i+1} - t_i) + (t - \eta(t)) W_{\eta(t)}$$

$$= \left(t_{\lfloor \frac{t}{\Delta t} \rfloor} W_{t_{\lfloor \frac{t}{\Delta t} \rfloor}} - t_0 W_0 \right) - \sum_{i=0}^{\lfloor \frac{t}{\Delta t} \rfloor - 1} t_{i+1} \left(W_{t_{i+1}} - W_{t_i} \right) + (t - \eta(t)) W_{\eta(t)}$$

$$= \eta(t) W_{\eta(t)} - \sum_{i=0}^{\lfloor \frac{t}{\Delta t} \rfloor - 1} t_{i+1} \left(W_{t_{i+1}} - W_{t_i} \right) + (t - \eta(t)) W_{\eta(t)} = t W_{\eta(t)} - \sum_{i=0}^{\lfloor \frac{t}{\Delta t} \rfloor - 1} t_{i+1} \left(W_{t_{i+1}} - W_{t_i} \right),$$

and by computing the variance of the previous sum:

$$\begin{aligned}
&\mathbb{V}\text{ar}\left[\int_0^t W_{\eta(s)}\,ds\right] = \mathbb{V}\text{ar}\left[tW_{\eta(t)} - \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\left(W_{t_{i+1}} - W_{t_i}\right)\right] \\
&= t^2\mathbb{V}\text{ar}\left[W_{\eta(t)}\right] + \mathbb{V}\text{ar}\left[\sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\left(W_{t_{i+1}} - W_{t_i}\right)\right] - 2t\sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\mathbb{C}\text{ov}\left[W_{\eta(t)}, W_{t_{i+1}} - W_{t_i}\right] \\
&= t^2\eta(t) + \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}^2\Delta t - 2t\sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\Delta t \\
&= t^2\eta(t) + (\Delta t)^3\sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1}(i+1)^2 - 2t(\Delta t)^2\sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1}(i+1) \\
&= t^2\eta(t) + (\Delta t)^3\frac{\lfloor\frac{t}{\Delta t}\rfloor\left(\lfloor\frac{t}{\Delta t}\rfloor+1\right)\left(2\lfloor\frac{t}{\Delta t}\rfloor+1\right)}{6} - 2t(\Delta t)^2\frac{\lfloor\frac{t}{\Delta t}\rfloor\left(\lfloor\frac{t}{\Delta t}\rfloor+1\right)}{2} \\
&= t\eta(t)(t-\eta(t)-\Delta t) + \frac{\eta(t)(\eta(t)+\Delta t)(2\eta(t)+\Delta t)}{6}.
\end{aligned}$$

Concerning the covariance:

$$\begin{aligned}
&\mathbb{C}\text{ov}\left[W_t, \int_0^t W_{\eta(s)}\,ds\right] = \mathbb{C}\text{ov}\left[W_t, tW_{\eta(t)} - \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\left(W_{t_{i+1}} - W_{t_i}\right)\right] \\
&= t\eta(t) - \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\Delta t = t\eta(t) - \frac{\eta(t)(\eta(t)+\Delta t)}{2}.
\end{aligned}$$

Finally, we obtain that:

$$\begin{bmatrix}\bar{Z}_t^{x,u}\\ \bar{V}_t^{x,u}\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}x+tu\\ u\end{bmatrix}, \sigma^2\begin{bmatrix} t\eta(t)(t-\eta(t)-\Delta t) + \frac{\eta(t)(\eta(t)+\Delta t)(2\eta(t)+\Delta t)}{6} & t\eta(t) - \frac{\eta(t)(\eta(t)+\Delta t)}{2} \\ t\eta(t) - \frac{\eta(t)(\eta(t)+\Delta t)}{2} & t\end{bmatrix}\right).$$

Let $t > \Delta t$ and $\Sigma_{t,\eta(t),\Delta t}$ denotes the above covariance matrix. When $t \neq \eta(t)$, we consider the index $k \in \mathbb{N}^*$ such that $\eta(t) = k\Delta t$ and $\varepsilon \in (0,1)$ such that $t = \varepsilon + k\Delta t$. It can be easily seen that:

$$\det(\Sigma_{t,\eta(t),\Delta t}) = \frac{1}{12}k\Delta t\left(12\varepsilon^3 + 12(k-1)\Delta t\varepsilon + 2(k-1)(2k-1)(\Delta t)^2\varepsilon + k(k-1)(\Delta t)^3\right).$$

So it can be seen that $\det(\Sigma_{t,\eta(t),\Delta t}) > 0$. Then, for $t = \eta(t) > \Delta t$ then:

$$\det(\Sigma_{t,\eta(t),\Delta t}) = \frac{\eta(t)^2}{12}\left(\eta(t)^2 - (\Delta t)^2\right) > 0.$$

Then, for any $t > \Delta t$, we have that the pdf of the r.v. $(\bar{Z}_t^{x,u}, \bar{V}_t^{x,u})$ is:

$$\bar{p}^L(0;x,u;t;\xi,\zeta) = \frac{1}{2\pi\sqrt{\det(\Sigma_{t,\eta(t),\Delta t})}}\exp\left(-\frac{1}{2}\begin{bmatrix}\xi-(x+tu)\\ \zeta-u\end{bmatrix}^T\Sigma_{t,\eta(t),\Delta t}\begin{bmatrix}\xi-(x+tu)\\ \zeta-u\end{bmatrix}\right).$$

When $t \leq \Delta t$, the position process is a degenerate random variable and the pdf becomes:

$$\bar{p}^L(0;x,u;\xi,\zeta) = \delta(\xi-(x+tu))\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{(\zeta-u)^2}{2\sigma^2 t}\right)$$

where δ is the Dirac delta distribution. $\square$

C Some complements to Section 6

For the sake of completeness, we present in this appendix the proofs of the section 6.

C.1 Proof of Theorem 6.1

We assume that the drift b is a $\mathcal{C}_b^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ function.

We consider the inputs (f_0, q) and assume the following

$$(H_{f_0,q}):\qquad f_0\in L^2(\mathcal{D}\times\mathbb{R}^d)\cap\mathcal{C}_b(\mathcal{D}\times\mathbb{R}^d)\text{ and }q\in L^2(\Sigma_T^+)\cap\mathcal{C}_b(\Sigma_T^+).$$

As a preliminary for the proof of Theorem 6.1, let us recall a more classical existence result for equation (6.5), issued from the application of Lions and Magenes Theorem A.3.

Lemma C.1. *Assume (H_{PDE}). Given two functions $f_0\in L^2(\mathcal{D}\times\mathbb{R}^d)$ and $q\in L^2(\Sigma_T^+)$, there exists a unique function f in $\mathcal{C}([0,T];L^2(\mathcal{D}\times\mathbb{R}^d))\cap\mathcal{H}(Q_T)$ admitting a trace $\gamma(f)\in L^2(\Sigma_T)$ along the boundary Σ_T, satisfying equation (6.5) in the sense that*

$$\begin{aligned}
&\partial_t f-(u\cdot\nabla_x f)-(b(x,u)\cdot\nabla_u f)-\frac{\sigma^2}{2}\triangle_u f=0,\ \ \textit{in }\mathcal{H}'(Q_T),\\
&f(t=0,x,u)=f_0(x,u),\ \ \textit{on }\mathcal{D}\times\mathbb{R}^d,\\
&\gamma(f)(t,x,u)=q(t,x,u),\ \ \textit{on }\Sigma_T^+.
\end{aligned}\tag{C.1}$$

In particular, for all $t\in(0,T)$,

$$\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}+\sigma^2\|\nabla_u f\|^2_{L^2(Q_t)}+\|\gamma(f)\|^2_{L^2(\Sigma_t^-)}\le C_{T,\|\nabla_u\cdot b\|_\infty,\sigma}\left(\|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}+\|q\|^2_{L^2(\Sigma_t^+)}\right)\tag{C.2}$$

where $C_{T,\|\nabla_u\cdot b\|_\infty,\sigma}$ is a positive constant depending on T, b and σ.

Proof. **Step 1: Construction of a solution in $\mathcal{H}(Q_T)$**

Let λ be a real to be defined later on and the functions $\bar f\colon(t,x,u)\in Q_T\mapsto\exp(-\lambda t)f(t,x,u)$ and $\bar q\colon(t,x,u)\in\Sigma_T^+\mapsto\exp(-\lambda t)q(t,x,u)$. Then (C.1) becomes:

$$\begin{aligned}
&\partial_t\bar f-(u\cdot\nabla_x\bar f)-(b(x,u)\cdot\nabla_u\bar f)-\frac{\sigma^2}{2}\triangle_u\bar f+\lambda\bar f=0,\ \ \text{in }\mathcal{H}'(Q_T),\\
&\bar f(t=0,x,u)=f_0(x,u),\ \ \text{on }\mathcal{D}\times\mathbb{R}^d,\\
&\gamma(\bar f)(t,x,u)=\bar q(t,x,u),\ \ \text{on }\Sigma_T^+.
\end{aligned}\tag{C.3}$$

In order to apply Theorem (A.3), the space E is identified as $\mathcal{H}(Q_T)$ considered with its norm. Also we define the space $F=\{\psi\in\mathcal{C}_c^\infty(\overline{Q}_T;\mathbb{R}),\text{ s.t. }\psi=0\text{ on }\{T\}\times\mathcal{D}\times\mathbb{R}^d\text{ and on }\overline{\Sigma_T^-}\}$ together with its norm:

$$|\psi|_F^2=\|\psi\|^2_{\mathcal{H}(Q_T)}+\|\psi\|^2_{L^2(\Sigma_T^+)}$$

which shows that the canonical injection from F into $\mathcal{H}(Q_T)$ is continuous.

The variational form of (C.3) is written as: for $\psi\in F$,

$$\begin{aligned}
&\int_{Q_T}\psi\partial_t\bar f-\int_{Q_T}\psi(u\cdot\nabla_x\bar f)-\int_{Q_T}\psi(b(x,u)\cdot\nabla_u\bar f)-\frac{\sigma^2}{2}\int_{Q_T}\psi\triangle_u\bar f+\lambda\int_{Q_T}\psi\bar f=0\\
\iff\quad&-\int_{Q_T}\bar f\partial_t\psi+\int_{Q_T}\bar f(u\cdot\nabla_x\psi)-\int_{Q_T}\psi(b(x,u)\cdot\nabla_u\bar f)+\frac{\sigma^2}{2}\int_{Q_T}\nabla_u\psi\cdot\nabla_u\bar f+\lambda\int_{Q_T}\psi\bar f\\
&=\int_{\Sigma_T}(u\cdot n_{\mathcal{D}})\psi\bar f-\int_{\mathcal{D}\times\mathbb{R}^d}\psi(T,\cdot,\cdot)\bar f+\int_{\mathcal{D}\times\mathbb{R}^d}\psi(0,\cdot,\cdot)f_0\\
&=-\int_{\Sigma_T^-}|(u\cdot n_{\mathcal{D}})|\psi\bar f+\int_{\Sigma_T^+}|(u\cdot n_{\mathcal{D}})|\psi\bar q-\int_{\mathcal{D}\times\mathbb{R}^d}\psi(T,\cdot,\cdot)\bar f+\int_{\mathcal{D}\times\mathbb{R}^d}\psi(0,\cdot,\cdot)f_0\\
&=\int_{\Sigma_T^+}|(u\cdot n_{\mathcal{D}})|\psi\bar q+\int_{\mathcal{D}\times\mathbb{R}^d}\psi(0,\cdot,\cdot)f_0
\end{aligned}$$

which allows to identify the bilinear form $A\colon(\varphi,\psi)\in(\mathcal{H}(Q_T)\times F)\mapsto A(\varphi,\psi)$ as:

$$A(\varphi,\psi)=-\int_{Q_T}\varphi\partial_t\psi+\int_{Q_T}\varphi(u\cdot\nabla_x\psi)-\int_{Q_T}(b(x,u)\cdot\nabla_u\varphi)\psi+\frac{\sigma^2}{2}\int_{Q_T}\nabla_u\varphi\cdot\nabla_u\psi+\lambda\int_{Q_T}\varphi\psi$$

and the linear form $L\colon \psi \in F \mapsto L(\psi)$:

$$L(\psi) = \int_{\Sigma_T^+} |(u \cdot n_{\mathcal{D}})| \bar{q} \psi + \int_{\mathcal{D} \times \mathbb{R}^d} f_0(0, \cdot, \cdot) \psi.$$

So the shorthand version of the variational form of (C.3) is:

$$A(\bar{f}, \psi) = L(\psi). \tag{C.4}$$

It is clear that the mapping $A(\cdot, \psi)$ from $\mathcal{H}(Q_T)$ into $\mathbb{R}$ is continuous for any ψ in F. Concerning the coercivity, for any ψ in F:

$$\begin{aligned}
A(\psi, \psi) &= -\int_{Q_T} \psi \partial_t \psi + \int_{Q_T} \psi (u \cdot \nabla_x \psi) - \int_{Q_T} (b(x,u) \cdot \nabla_u \psi) \psi + \frac{\sigma^2}{2} \int_{Q_T} \nabla_u \psi \cdot \nabla_u \psi + \lambda \int_{Q_T} \psi^2 \\
&= -\frac{1}{2} \|\psi(T, \cdot, \cdot)\|^2_{L^2(\mathcal{D} \times \mathbb{R}^d)} + \frac{1}{2} \|\psi(0, \cdot, \cdot)\|^2_{L^2(\mathcal{D} \times \mathbb{R}^d)} + \frac{1}{2} \int_{Q_T} u \cdot \nabla_x \psi^2 - \frac{1}{2} \int_{Q_T} b(x,u) \cdot \nabla_u \psi^2 \\
&\quad + \frac{\sigma^2}{2} \|\nabla_u \psi\|^2_{L^2(Q_T)} + \lambda \|\psi\|^2_{L^2(Q_T)} \\
&= \frac{1}{2} \|\psi(0, \cdot, \cdot)\|^2_{L^2(\mathcal{D} \times \mathbb{R}^d)} + \frac{1}{2} \int_{\Sigma_T^+} |(u \cdot n_{\mathcal{D}})| \psi^2 + \frac{1}{2} \int_{Q_T} (\nabla_u \cdot b(x,u)) \psi^2 + \frac{\sigma^2}{2} \|\nabla_u \psi\|^2_{L^2(Q_T)} + \lambda \|\psi\|^2_{L^2(Q_T)} \\
&\geq \frac{1}{2} \|\psi\|^2_{L^2(\Sigma_T^+)} + \left(\lambda - \frac{1}{2} \|\nabla_u \cdot b(x,u)\|_{L^\infty(Q_T)}\right) \|\psi\|^2_{L^2(Q_T)} + \frac{\sigma^2}{2} \|\nabla_u \psi\|^2_{L^2(Q_T)} \\
&\geq \min\left(\lambda - \frac{1}{2} \|\nabla_u \cdot b(x,u)\|_{L^\infty(Q_T)}, \frac{\sigma^2}{2}, \frac{1}{2}\right) |\psi|^2_F.
\end{aligned}$$

By choosing $\lambda > \frac{1}{2} \|\nabla_u \cdot b(x,u)\|_{L^\infty(Q_T)}$, A becomes a coercive application on $F \times F$ and, as such, by Theorem (A.3), there exists $\bar{f}$ in $\mathcal{H}(Q_t)$ such that for any ψ in F, the equation (C.4) is satisfied. Multiplying this function by $\exp(\lambda t)$ gives the desired result.

Step 2: Existence of the trace on Σ_T and proof of energy inequality

Consider now the transport operator $\mathcal{T} = \partial_t - u \cdot \nabla_x$ and the spaces:

$$\mathcal{Y}(Q_T) = \{\varphi \in \mathcal{H}(Q_T); -\mathcal{T}(\varphi) \in \mathcal{H}'(Q_T)\}$$

and

$$\mathcal{V}(Q_T) = \{\psi \in \mathcal{H}(Q_T); \psi \text{ has traces } \gamma(\psi^\pm) \text{ on } \Sigma_T^\pm, \gamma(\psi^\pm) \in L^2(\Sigma_T^\pm)\}.$$

We shall show that $f \in \mathcal{Y}(Q_T)$. Let $\varphi \in \mathcal{C}_c^\infty(Q_T)$, then:

$$\begin{aligned}
\left| \int_{Q_T} \mathcal{T}(f) \varphi \right| &= \left| \int_{Q_T} -\varphi b \cdot \nabla_u f - \frac{\sigma^2}{2} \int_{Q_T} \triangle_u f \varphi \right| \\
&\leq \|b\|_{L^\infty(\mathcal{D} \times \mathbb{R}^d)} \int_{Q_T} |\nabla_u f| |\varphi| + \left| \int_{Q_T} \nabla_u f \nabla_u \varphi \right| \\
&\leq \max\left\{ \|b\|_{L^\infty(\mathcal{D} \times \mathbb{R}^d)}, 1 \right\} \|f\|_{\mathcal{H}(Q_T)} \|\varphi\|_{\mathcal{H}(Q_T)}.
\end{aligned}$$

This means that $f \in \mathcal{Y}(Q_T)$, so by [10], f admits a trace on the border of the domain $\mathcal{D}$ and $f \in \mathcal{V}(Q_t)$, and the Green formula (A.4) can be applied. Equation (C.1) can be rewritten as:

$$\mathcal{T}(f)(t,x,u) - b(x,u) \cdot \nabla_u f(t,x,u) - \frac{\sigma^2}{2} \triangle_u f = 0$$

and by multiplying with f and integrating over Q_T, we obtain that:

$$\begin{aligned}
&(\mathcal{T}(f),f)_{\mathcal{H}'(Q_T),\mathcal{H}(Q_T)} - \int_{Q_T}(b\cdot\nabla_u f)f - \frac{\sigma^2}{2}\int_{Q_T} f\triangle_u f = 0\\
\iff\ &(\mathcal{T}^*(f),f)_{\mathcal{H}'(Q_T),\mathcal{H}(Q_T)} + \int_{\mathcal{D}\times\mathbb{R}^d} f^2(T,x,u)\,dxdu - \int_{\mathcal{D}\times\mathbb{R}^d} f_0^2\\
&\quad - \int_{\Sigma_T^+}(u\cdot n_{\mathcal{D}})q^2 - \int_{\Sigma_T^-}(u\cdot n_{\mathcal{D}})\gamma^-(f)^2 - \int_{Q_T}(b\cdot\nabla_u f)f - \frac{\sigma^2}{2}\int_{Q_T} f\triangle_u f = 0
\end{aligned}$$

Since $\mathcal{T}^* = -\mathcal{T}$, we add the two previous equations to obtain that:

$$\begin{aligned}
&\int_{\mathcal{D}\times\mathbb{R}^d} f^2(T,x,u)\,dxdu - \int_{\mathcal{D}\times\mathbb{R}^d} f_0^2 - \int_{\Sigma_T^+}(u\cdot n_{\mathcal{D}})q^2\\
&\quad - \int_{\Sigma_T^-}(u\cdot n_{\mathcal{D}})\gamma^-(f)^2 - \int_{Q_T} 2(b\cdot\nabla_u f)f - \sigma^2\int_{Q_T} f\triangle_u f = 0
\end{aligned}$$

As T is arbitrary, this also writes for any $t\le T$,

$$\begin{aligned}
&\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \int_{\Sigma_t^-}|u\cdot n_{\mathcal{D}}|(\gamma(f)^-)^2 + \int_{Q_t}(\nabla_u\cdot b)f^2 + \sigma^2\|\nabla_u f\|^2_{L^2(Q_t,\mathbb{R}^d)}\\
&= \|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \int_{\Sigma_t^+}|u\cdot n_{\mathcal{D}}|(\gamma(f)^+)^2
\end{aligned}$$

and

$$\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\gamma(f)\|^2_{L^2(\Sigma_t^-)} + \sigma^2\|\nabla_u f\|^2_{L^2(Q_t)} = -\int_{Q_t}(\nabla_u\cdot b)f^2 + \|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)}\,.$$

From this, one has the inequality:

$$\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} \le \|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)} + \|\nabla_u\cdot b\|_{L^\infty(Q_t)}\int_0^t ds\,\|f(s)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}$$

So by Grownall's lemma:

$$\begin{aligned}
&\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} \le \left(\|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)}\right)\exp\left(\frac{1}{2}\|\nabla_u\cdot b\|_{L^\infty(Q_t)}\,t\right)\\
\text{and}\quad &\|f\|^2_{L^2(Q_t)} \le t\left(\|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)}\right)\exp\left(\frac{1}{2}\|\nabla_u\cdot b\|_{L^\infty(Q_t)}\,t\right)
\end{aligned}$$

which when plugged in the previous equality, allows to obtain (C.2) with

$$C_{T,\|\nabla_u\cdot b\|_\infty,\sigma} = 1 + T\|\nabla_u\cdot b\|_{L^\infty(Q_T)}\exp\left(\|\nabla_u\cdot b\|_{L^\infty(Q_T)}\,T\right).$$

The uniqueness of the solution is obtained from the energy inequality and linearity of the equation. □

Proposition C.2 (Interior regularity)**.** *Under ($H_{f_0,q}$), the unique solution f of (C.1) belongs to $\mathcal{C}^{1,1,2}(Q_T)$.*

Proof. To prove this proposition, it is sufficient to show that, for all $z_0 := (t_0,x_0,u_0)$ in Q_T, there exists $r>0$ such that f belongs to $\mathcal{C}^{1,1,2}(B_{z_0}(r))$ where $B_{z_0}(r)\subset Q_T$ is the open ball centred at z_0 of radius r. To this end, we use the Sobolev embeddings (see e.g. [9], Corollary 9.15): for $m = \lfloor d/2\rfloor + 2 - \lfloor 1-(d/2-\lfloor d/2\rfloor)\rfloor$, we have[2]

$$W^{2,2}((0,T))\subset\mathcal{C}^1([0,T]),\quad W^{m,2}(B_{x_0}(r))\subset\mathcal{C}^1(\overline{B_{x_0}(r)}),\quad W^{m+1,2}(B_{u_0}(r))\subset\mathcal{C}^2(\overline{B_{u_0}(r)}).$$

We thus first prove that for some $r>0$,

$$\|\partial_t^2 f\|_{L^2(B_{z_0}(r))} + \sum_{\eta\in\mathbb{N}^d;|\eta|\le m}\|D_x^\eta f\|_{L^2(B_{z_0}(r))} + \sum_{\kappa\in\mathbb{N}^d;|\kappa|\le m+1}\|D_u^\kappa f\|_{L^2(B_{z_0}(r))} < +\infty, \tag{C.5}$$

[2]For $\lfloor x\rfloor$ the nearest integer lower than $x\in\mathbb{R}^+$.

where D_x^η and D_u^κ refer to the differential operators given by

$$D_x^\eta f = \partial_{x_1}^{\eta_1}\partial_{x_2}^{\eta_2}\cdots\partial_{x_d}^{\eta_d} f, \text{ for } \eta = (\eta_1,\eta_2,\cdots,\eta_d)\in\mathbb{N}^d,$$
$$D_u^\kappa f = \partial_{u_1}^{\kappa_1}\partial_{u_2}^{\kappa_2}\cdots\partial_{u_d}^{\kappa_d} f, \text{ for } \kappa = (\kappa_1,\kappa_2,\cdots,\kappa_d)\in\mathbb{N}^d.$$

Since b is assumed to be a $\mathcal{C}_b^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ function here, we can iterate the whole argument and prove (C.5) for higher order of Sobolev derivatives to conclude that f belongs to $\mathcal{C}^{1,1,2}(B_{z_0}(r))$.

The proof of (C.5), is based on a bootstrap argument that uses the regularity results (in fractional Sobolev spaces) obtained in Bouchut [7] for the solution to kinetic equation (see Theorem A.5).

Step 1. Let us start with the regularity along the (x,u)-variables. We proceed by induction on a truncated version of f.

For any $r_0>0$ such that $B_{z_0}(r_0)\subsetneq Q_T$, we denote by $\beta_{r_0}: Q_T\to[0,1]$, a $\mathcal{C}_c^\infty(Q_T)$-cutoff function such that

$$\begin{cases} \beta_{r_0} = 1 \text{ on } \overline{B_{z_0}(\frac{r_0}{2})},\\ \beta_{r_0} = 0 \text{ on } Q_T\setminus B_{z_0}(r_0).\end{cases}$$

We further assume that there exists a constant C depending on r_0 such that

$$\sum_{\eta\in\mathbb{N}^d;|\eta|\le m+1;\,\beta\in\mathbb{N}^d;|\beta|\le m+2}\|\partial_t^2 D_x^\eta D_u^\beta\beta_{r_0}\|_{L^\infty(Q_T)}\le C.$$

Starting from $f\in L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ given in Lemma C.1, the truncated function $f_{r_0}:=\beta_{r_0}f$ satisfies, in the sense of distributions,

$$\begin{cases}\partial_t f_{r_0} - (u\cdot\nabla_x f_{r_0}) - \dfrac{\sigma^2}{2}\triangle_u f_{r_0} = \Gamma_{r_0}f + (\Psi_{r_0}\cdot\nabla_u f), \text{ on } Q_T,\\ f_{r_0}|_{t=0}=0, \text{ on } \mathcal{D}\times\mathbb{R}^d,\\ \gamma^\pm(f_{r_0})=0, \text{ on } \Sigma_T^\pm,\end{cases}$$

with $\Gamma_{r_0} := \partial_t\beta_{r_0} - (u\cdot\nabla_u\beta_{r_0}) - \frac{\sigma^2}{2}\triangle_u\beta_{r_0}$ and $\Psi_{r_0} := -\sigma^2\nabla_u\beta_{r_0} + (\beta_{r_0}b)$. Extending f_{r_0}, $\Gamma_{r_0}f$ and $(\Psi_{r_0}\cdot\nabla_u f)$ on the whole space $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ by 0 outside $B_{z_0}(r_0)$, one has

$$\partial_t f_{r_0} - (u\cdot\nabla_x f_{r_0}) - \frac{\sigma^2}{2}\triangle_u f_{r_0} = g_{r_0}, \text{ in } (\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))' \tag{C.6}$$

where $g_{r_0}:=\Gamma_{r_0}f + (\Psi_{r_0}\cdot\nabla_u f)$. Let us now recall Theorem 1.5 (and its proof) in [7]: for $\alpha\in(0,1)$, we further denote by D_x^α the fractional derivative w.r.t. x-variables, defined as the fractional Laplace operator of order α

$$D_x^\alpha = (-\triangle_x)^{\alpha/2}.$$

Since $\Gamma_{r_0}f$ and $(\Psi_{r_0}\cdot\nabla_u f)$ are in $L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$, Theorem A.5-(b) implies that $D_x^{2/3}f_{r_0}$, $|\nabla_u D_x^{1/3}f_{r_0}|$, and $\triangle_u f_{r_0}$ are in $L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$. As $\beta_{r_0}=1$ on $B_{z_0}(\frac{r_0}{2})$, this particularly ensures that

$$\|D_x^{2/3}f\|_{L^2(B_{z_0}(\frac{r_0}{2}))} = \|D_x^{2/3}f_{r_0}\|_{L^2(B_{z_0}(\frac{r_0}{2}))}\le\|D_x^{2/3}f_{r_0}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}<+\infty,$$
$$\|\nabla_u D_x^{1/3}f\|_{L^2(B_{z_0}(\frac{r_0}{2}))} = \|\nabla_u D_x^{1/3}f_{r_0}\|_{L^2(B_{z_0}(\frac{r_0}{2}))}\le\|\nabla_u D_x^{1/3}f_{r_0}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}<+\infty.$$

By setting $r_1:=\frac{r_0}{2}$ and $f_{r_1}:=\beta_{r_1}f$, it follows that $D_x^{1/3}f_{r_1}\in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$ (since[3] $\|D_x^{1/3}f_{r_1}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}\le\|f_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}\|D_x^{2/3}f_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}$.) Furthermore, as we are dealing with L^2 norm, the fractional Sobolev space H^α, $0<\alpha<1$ and the fractional Laplacian operator D^α are connected and (see [22], Proposition 3.6), $\|f\|_{H^s}=C\|D^\alpha f\|_{L^2}$ for C a dimensional constant. Moreover, as g_{r1} is the product of $\mathcal{C}_c^\infty$ functions with f_{r1} and $\nabla_u f_{r1}$, we can apply the Lemma 5.3 in [22], to get

$$\|D_x^{1/3}g_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} = C\|g_{r1}\|_{H^{1/3}(B_{z_0}(r_1))}\le C'\|f\|_{H^{1/3}(B_{z_0}(r_1))} + C'\|\nabla_u f\|_{H^{1/3}(B_{z_0}(r_1))}<\infty.$$

[3]This can be shown by applying a Cauchy-Schwarz inequality in the alternative definition of the fractional derivative in L^2 via Fourier transform, see e.g. [22].

Applying the differential operator $D_x^{1/3}$ to (C.6), one can check that $D_x^{1/3}f_{r_1}$ satisfies

$$\partial_t D_x^{1/3}f_{r_1}-(u\cdot\nabla_x D_x^{1/3}f_{r_1})-\frac{\sigma^2}{2}\triangle_u D_x^{1/3}f_{r_1}=D_x^{1/3}g_{r_1},\ \text{ in }(\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))'. \tag{C.7}$$

From Theorem A.5-(b) again, we obtain that $|\nabla_x f_{r_1}|\leq C|D_x^{2/3}(D_x^{1/3}f_{r_1})|\in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$, and $|\nabla_u D_x^{2/3}f_{r_1}|\in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$. Therefore, $|\nabla_x f|\in L^2(B_{z_0}(\frac{r_1}{2}))$. Applying again $D_x^{1/3}$ to (C.7), applying Theorem A.5-(b) a third time, one can also deduce that $|\nabla_u\nabla_x f|$ is in $L^2(B_{z_0}(\frac{r_0}{2^3}))$.

We obtain the regularity w.r.t. u by applying the differential operator ∂_{u_i} to Eq. (C.6). Hence $\partial_{u_i}f_{r_1}$ satisfies

$$\partial_t\partial_{u_i}f_{r_1}-(u\cdot\nabla_x\partial_{u_i}f_{r_1})-\frac{\sigma^2}{2}\triangle_u\partial_{u_i}f_{r_1}=\partial_{u_i}g_{r_1}+\partial_{x_i}f_{r_1},\ \text{ in }(\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))', \tag{C.8}$$

where

$$\partial_{u_i}g_{r_1}=(\partial_{u_i}\Gamma_{r_1})f+\Gamma_{r_1}\partial_{u_i}f+(\Psi_{r_1}\cdot\nabla_u\partial_{u_i}f)+(\partial_{u_i}\Psi_{r_1}\cdot\nabla_u f)\,.$$

Theorem A.5-(a) ensures that $\|\triangle_u f_{r_0}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}<+\infty$. As f_{r_0} has a compact support, standard arguments give that

$$\sum_{1\leq i,j\leq d}\|\partial^2_{u_i,u_j}f_{r_0}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}=\|\triangle_u f_{r_0}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}<+\infty$$

and thus

$$\sum_{1\leq i,j\leq d}\|\partial^2_{u_i,u_j}f\|^2_{L^2(B_{z_0}(r_1))}=\sum_{1\leq i,j\leq d}\|\partial^2_{u_i,u_j}f_{r_0}\|_{L^2(B_{z_0}(r_1))}<+\infty.$$

Now we set $h=\partial_{u_i}g_{r_1}+\partial_{x_i}f_{r_1}$ with $\|h\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}\leq\|\nabla_u g_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}+\|\nabla_x f_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}<+\infty$, since

$$\|\nabla_u g_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}\leq C\big(\|f\|_{L^2(B_{z_0}(r_1))}+\|\nabla_u f\|_{L^2(B_{z_0}(r_1))}+\sum_{1\leq i,j\leq d}\|\partial^2_{u_i,u_j}f\|^2_{L^2(B_{z_0}(r_1))}\big)<+\infty.$$

Theorem A.5-(a) ensures that $|\nabla_u(\triangle_u f_{r_1})|\in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$ and hence that $|\triangle_u\nabla_u f|\in L^2(B_{z_0}(\frac{r_1}{2}))$.

We sum up the estimations we have obtained as

$$\|\nabla_x f\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))}+\|\nabla_x\nabla_u f\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))}+\|\triangle_u\nabla_u f\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))}<+\infty. \tag{C.9}$$

We extend (C.9) to higher order differentials through the following induction argument: we have proved that for $N=1$,

$$D_x^\eta f,\ |\nabla_u D_x^\eta f|,\ |\nabla_x D_u^{\eta'}f|,\ |\nabla_u D_u^\eta f|\text{ are all in }L^2(B_{z_0}(R_N)),\ \text{ for all }\ \eta\in\mathbb{N}^d\text{ such that }1\leq|\eta|\leq N,$$

with $R_N=r_0/2^{3N}$ and $\eta'\in\mathbb{N}^d$ is such that $|\eta'|=|\eta|-1$.

Starting from the induction assumption that $\|D_x^\eta f\|_{L^2(B_{z_0}(R_N))}+\|\nabla_u D_x^\eta f\|_{L^2(B_{z_0}(R_N))}<+\infty$, for $|\eta|\leq N$, we have that $D_x^\eta f_{R_N}$ satisfies

$$\partial_t D_x^\eta f_{R_N}-(u\cdot\nabla_x D_x^\eta f_{R_N})-\frac{\sigma^2}{2}\triangle_u D_x^\eta f_{R_N}=D_x^\eta g_{R_N},\ \text{ in }(\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))'.$$

Applying three times Theorem A.5-(b), we deduce as before that $|\nabla_x D_x^\eta f|$ and $|\nabla_x\nabla_u D_x^\eta f|$ are in $L^2(B_{z_0}(\frac{R_N}{2^3}))$.

Now, from the induction assumption $\|\nabla_u D_u^\eta f\|_{L^2(B_{z_0}(R_N))}+\|\nabla_x D_u^{\eta'}f\|_{L^2(B_{z_0}(R_N))}<+\infty$, for η and η', $|\eta|\leq N$, we have that $D_u^\eta f_{R_N}$ satisfies

$$\partial_t D_u^\eta f_{R_N}-(u\cdot\nabla_x D_u^\eta f_{R_N})-\frac{\sigma^2}{2}\triangle_u D_u^\eta f_{R_N}=D_u^\eta g_{R_N}+(D_u^\eta(u\cdot\nabla_x f_{R_N})-(u\cdot\nabla_x D_u^\eta f_{R_N}))\,,$$

in $(\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))'$. Since

$$\|D_u^\eta(u\cdot\nabla_x f_{R_N})-(u\cdot\nabla_x D_u^\eta f_{R_N})\|\leq\sum_{\eta';|\eta'|=N-1}\|\nabla_x D_u^{\eta'}f\|_{L^2(B_{z_0}(R_N))}<+\infty,$$

applying Theorem A.5-(a), we deduce as before that $\triangle_u D_u^\eta f\in L^2(B_{z_0}(\frac{R_N}{2}))$, which ensures that $\|\nabla_u D_u^\eta f|\in L^2(B_{z_0}(\frac{R_N}{2}))$. By applying Theorem A.5-(b) three times, we obtain that $|\nabla_x D_u^\eta f|\in L^2(B_{z_0}(\frac{R_N}{2^3}))$. This ends the proof of the induction $N+1$.

We iterate m times this induction and conclude that, for $r:=\frac{r_0}{2^{3m}}$,

$$\sum_{\eta\in\mathbb{N}^d;|\eta|\leq m}\|D_x^\eta f\|_{L^2(B_{z_0}(r))}+\sum_{\kappa\in\mathbb{N}^d;|\kappa|\leq m+1}\|D_u^\kappa f\|_{L^2(B_{z_0}(r))}<+\infty.$$

Step 2. Finally, we estimate $\|\partial_t^2 f\|_{L^2(B_{z_0}(r))}$. Since $\nabla_x f$ and $g_{\frac{r_0}{2^3}}$ are in $L^2(B_{z_0}(\frac{r_0}{2^3}))$, according to Theorem A.5-(a), we have

$$\begin{aligned}\|\partial_t f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} &\leq \|\partial_t f_{\frac{r_0}{2^3}} + (u\cdot\nabla_x f_{\frac{r_0}{2^3}})\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \|(u\cdot\nabla_x f_{\frac{r_0}{2^3}})\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))}\\ &\leq C\|g_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \frac{r_0}{2^3}\|\nabla_x f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} < +\infty.\end{aligned}$$

Moreover $\partial_{u_i} f_{\frac{r_0}{2^3}}$ satisfies (C.8) and $\partial_{x_i} f_{\frac{r_0}{2^3}}$ satisfies

$$\partial_t\partial_{x_i} f_{\frac{r_0}{2^3}} - (u\cdot\nabla_x\partial_{x_i} f_{\frac{r_0}{2^3}}) - \frac{\sigma^2}{2}\triangle_u\partial_{x_i} f_{\frac{r_0}{2^3}} = \partial_{x_i} g_{\frac{r_0}{2^3}} \text{ in } (\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))',$$

with $|\nabla_x f_{\frac{r_0}{2^3}}|$, $|\nabla_u f_{\frac{r_0}{2^3}}|$, $\partial^2_{x_i,x_j} f_{\frac{r_0}{2^3}}$, $\partial^2_{u_i,u_j} f_{\frac{r_0}{2^3}}$ and $\partial^2_{x_i,u_j} f_{\frac{r_0}{2^3}}$ in $L^2(B_{z_0}(\frac{r_0}{2^3}))$ for $1\leq i,j\leq d$. We easily deduce that $\partial_{x_i} g_{\frac{r_0}{2^3}}$ and $\partial_{u_i} g_{\frac{r_0}{2^3}}$ are also in $L^2(B_{z_0}(\frac{r_0}{2^3}))$. From Theorem A.5-(a) again it follows that

$$\begin{aligned}\|\partial_t\partial_{x_i} f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} &\leq C\|\partial_{x_i} g_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \frac{r_0}{2^3}\|\nabla_x\partial_{x_i} f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))},\\ \|\partial_t\partial_{u_i} f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} &\leq C\|\partial_{u_i} g_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \frac{r_0}{2^3}\|\nabla_x\partial_{u_i} f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))}.\end{aligned}$$

so that $|\partial_t\nabla_x f_{\frac{r_0}{2^3}}|$ and $|\partial_t\nabla_u f_{\frac{r_0}{2^3}}|$ are in $L^2(B_{z_0}(\frac{r_0}{2^3}))$. Now we observe that $\partial_t f_{\frac{r_0}{2^3}}$ satisfies

$$\partial_t^2 f_{\frac{r_0}{2^3}} - (u\cdot\nabla_x\partial_t f_{\frac{r_0}{2^3}}) - \frac{\sigma^2}{2}\triangle_u\partial_t f_{\frac{r_0}{2^3}} = \partial_t g_{\frac{r_0}{2^3}} \text{ in } (\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))',$$

with

$$\partial_t g_{\frac{r_0}{2^3}} = \Gamma_{\frac{r_0}{2^3}}\partial_t f + (\Psi_{\frac{r_0}{2^3}}\cdot\nabla_u\partial_t f) + (\partial_t\Gamma_{\frac{r_0}{2^3}})f + \left(\partial_t\Psi_{\frac{r_0}{2^3}}\cdot\nabla_u f\right) \in L^2(B_{z_0}(\frac{r_0}{2^3})).$$

It follows that $\partial_t^2 f\in L^2(B_{z_0}(\frac{r_0}{2^4}))$ since

$$\|\partial_t^2 f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} \leq C\|\partial_t g_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \frac{r_0}{2^3}\|\nabla_x(\partial_t f)\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} < +\infty.$$

This enables us to conclude on (C.5).

□

Lemma C.3. *Let f be given as in Lemma C.1. Then for a.e. $(t,x,u)\in Q_T$,*

$$\inf_{(x,u)\in\mathcal{D}\times\mathbb{R}^d} f_0 \wedge \inf_{(t,x,u)\in\Sigma_T-\Sigma_T^0} q(t,x,u) \leq f(t,x,u) \leq \sup_{(x,u)\in\mathcal{D}\times\mathbb{R}^d} f_0 \vee \sup_{(t,x,u)\in\Sigma_T-\Sigma_T^0} q(t,x,u).$$

Proof. Let $\{\eta_R\}_{R>0}$ be a sequence of $\mathcal{C}^\infty$-cutoff functions on $\mathbb{R}^d$ such that, for all $R>0$, $\eta_R=\eta_R(u)\in L^1\cap L^\infty(\mathbb{R}^d)$ and there exists $0<C_R<\infty$ such that

$$|\nabla_u\eta_R(u)| + |\triangle_u\eta_R(u)| \leq C_R\eta_R(u),\ \forall\, u\in\mathbb{R}^d,$$

(for instance, take $\eta_R(u)=R^2/(R^2+|u|^2)$). Taking $\lambda_\kappa(t)=\exp\{\kappa t\}$ for κ a real number that will be chosen later and

$$M = \sup_{(x,u)\in\mathcal{D}\times\mathbb{R}^d} f_0 \vee \sup_{(t,x,u)\in\Sigma_T-\Sigma_T^0} q(t,x,u),$$

we get that

$$\begin{aligned}&L(\eta_R\lambda_\kappa\left|(f-M)^+\right|^2)\\ &= \left|(f-M)^+\right|^2 L(\eta_R\lambda_\kappa) + \eta_R\lambda_\kappa L(\left|(f-M)^+\right|^2) - \sigma^2\lambda_\kappa\left(\nabla_u\eta_R\cdot\nabla_u\left|(f-M)^+\right|^2\right).\end{aligned}\tag{C.10}$$

Let us point out that the function $\triangle_u|(f-M)^+|^2$ is well defined a.e. on Q_T since, using Theorem A.2, one can check that $\triangle_u|(f-M)^+|^2 = 2\nabla_u\cdot((f-M)^+\nabla_u(f-M)) = 2((f-M)^+\triangle_u(f-M)) + 2\left|\nabla_u(f-M)\right|^2\mathbb{1}_{\{f>M\}}$. In particular

$$\begin{aligned}&L(\left|(f-M)^+\right|^2)\\ &= 2\left(\partial_t(f-M) - u\cdot\nabla_x(f-M) - b\cdot_u(f-M) - \frac{\sigma^2}{2}\triangle_u(f-M))\right)(f-M)^+ - \sigma^2\left|\nabla_u(f-M)\right|^2\\ &\leq 0,\end{aligned}$$

Therefore, integrating (C.10) over Q_T, we have

$$\begin{aligned}
&\int_{Q_T} L(\eta_R\lambda_\kappa \left|(f-M)^+\right|^2) \\
&= \int_{Q_T} \left|(f-M)^+\right|^2 L(\eta_R\lambda_\kappa) + \eta_R\lambda_\kappa L(\left|(f-M)^+\right|^2) - \sigma^2\lambda_\kappa \left(\nabla_u\eta_R \cdot \nabla_u \left|(f-M)^+\right|^2\right) \\
&\leq \int_{Q_T} \left|(f-M)^+\right|^2 L(\eta_R\lambda_\kappa) - \sigma^2\lambda_\kappa \left(\nabla_u\eta_R \cdot \nabla_u \left|(f-M)^+\right|^2\right)
\end{aligned} \tag{C.11}$$

Observing that an integration by part on the second integral on the right-hand side of (C.11) gives

$$\begin{aligned}
&\int_{Q_T} \left|(f-M)^+\right|^2 L(\eta_R\lambda_\kappa) - \sigma^2\lambda_\kappa \left(\nabla_u\eta_R \cdot \nabla_u \left|(f-M)^+\right|^2\right) \\
&= \int_{Q_T} \left(L(\eta_R\lambda_\kappa) + \sigma^2\lambda_\kappa \triangle_u \eta_R\right) \left|(f-M)^+\right|^2
\end{aligned}$$

Using an integration by part for the left-hand side of (C.11) and, since

$$|(f_0-M)^+| = 0 \text{ on } \mathcal{D}\times\mathbb{R}^d, \ \ |(q-M)^+| = 0 \text{ on } \Sigma_T^+.$$

we get

$$\begin{aligned}
\int_{Q_T} L(\eta_R\lambda_\kappa \left|(f-M)^+\right|^2) = &\int_{\mathcal{D}\times\mathbb{R}^d} \eta_R\lambda_\kappa(T) \left|(f(T)-M)^+\right|^2 \\
&- \int_{\Sigma_T^-} (u\cdot n_{\mathcal{D}}(x))\eta_R\lambda_\kappa \left|(\gamma(f)-M)^+\right|^2 + \int_{Q_T} (\nabla_u \cdot b)\,\eta_R\lambda_\kappa \left|(f-M)^+\right|^2
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\mathcal{D}\times\mathbb{R}^d} \eta_R\lambda_\kappa(T) \left|(f(T)-M)^+\right|^2 - \int_{\Sigma_T^-} (u\cdot n_{\mathcal{D}}(x))\eta_R\lambda_\kappa \left|(\gamma(f)-M)^+\right|^2 \\
&= \int_{Q_T} \left(L(\eta_R\lambda_\kappa) + \sigma^2\lambda_\kappa\triangle_u\eta_R - (\nabla_u\cdot b)\,\eta_R\lambda_\kappa\right) \left|(f-M)^+\right|^2 \\
&= \int_{Q_T} \left(\kappa\eta_R - \nabla_u\eta_R\cdot b + \frac{\sigma^2}{2}\triangle_u\eta_R - (\nabla_u\cdot b)\,\eta_R\right)\lambda_\kappa \left|(f-M)^+\right|^2 \\
&\leq \int_{Q_T} \left(\kappa + C_R(1+\frac{\sigma^2}{2} + \|b\|_{L^\infty}) + \|\nabla_u\cdot b\|_{L^\infty}\right)\eta_R\lambda_\kappa \left|(f-M)^+\right|^2.
\end{aligned}$$

Since

$$\int_{\mathcal{D}\times\mathbb{R}^d} \eta_R\lambda_\kappa(T) \left|(f(T)-M)^+\right|^2 - \int_{\Sigma_T^-} (u\cdot n_{\mathcal{D}}(x))\eta_R\lambda_\kappa \left|(\gamma(f)-M)^+\right|^2 \geq 0,$$

choosing $\kappa < 0$ such that

$$\kappa + C_R(1+\frac{\sigma^2}{2} + \|b\|_{L^\infty}) + \|\nabla_u\cdot b\|_{L^\infty} < 0$$

implies that $(f-M)^+ = 0$ and that $f \leq M$ a.e. on Q_T. Replacing $f-M$ by $m-f$, for

$$m = \inf_{(x,u)\in\mathcal{D}\times\mathbb{R}^d} f_0 \wedge \inf_{(t,x,u)\in\Sigma_T-\Sigma_T^0} q(t,x,u),$$

and using similar arguments yields to $f \geq m$ a.e. on Q_T. □

Proposition C.4 (Continuity up to Σ_T^+). *Assume (H_{PDE}) and ($H_{f_0,q}$). Let $f \in \mathcal{C}^{1,1,2}(Q_T)\cap\mathcal{C}([0,T];L^2(\mathcal{D}\times\mathbb{R}^d))\cap\mathcal{H}(Q_T)$ be the solution to (6.5) with inputs (f_0,q). Then f is continuous up to Σ_T^+.*

Proof. To show the continuity up to the boundary Σ_T^+, we follow the classical method of local barrier functions (see e.g. [15]). Let $(t_0, x_0, u_0) \in \Sigma_T^+$ (i.e. $t_0 \in (0,T), x_0^{(d)} = 0, (u \cdot n_{\mathcal{D}}(x)) = -u_0^{(d)} > 0$). Since q is continuous in Σ_T^+, we can assume that for any $\epsilon > 0$, there exists a neighborhood $\mathcal{O}^\epsilon_{t_0,x_0,u_0}$ such that

$$q(t_0, x_0, u_0) - \epsilon \le q(t, x, u) \le q(t_0, x_0, u_0) + \epsilon, \ \ \forall (t, x, u) \in \mathcal{O}^\epsilon_{t_0,x_0,u_0} \cap \Sigma_T^+.$$

In addition, since $u_0 \cdot n_{\mathcal{D}}(x_0) = -u_0^{(d)} > 0$, by reducing $\mathcal{O}^\epsilon_{t_0,x_0,u_0}$, we can assume that $u \cdot n_{\mathcal{D}}(x) = -u^{(d)} > \eta > 0$ for all $(t, x, u) \in \mathcal{O}^\epsilon_{t_0,x_0,u_0}$. Consequently, by setting $\varrho(x)\colon x \in \mathbb{R}^d \mapsto \mathrm{dist}(x, \partial\mathcal{D})$ (which is simply $\varrho(x) = x^{(d)}$), and

$$L := \partial_t - (u \cdot \nabla_x) - (b(x, u) \cdot \nabla_u) - \frac{\sigma^2}{2}\triangle_u,$$

we observe that, for all $(t, x, u) \in \mathcal{O}^\epsilon_{t_0,x_0,u_0}$,

$$L(\varrho)(t, x, u) = -(u \cdot \nabla\varrho(x)) = u \cdot n_{\mathcal{D}}(x) > \eta > 0. \tag{C.12}$$

Reducing again $\mathcal{O}^\epsilon_{t_0,x_0,u_0}$, we can assume that $\mathcal{O}^\epsilon_{t_0,x_0,u_0}$ has the form $(t_0 - \delta_\epsilon, t_0 + \delta_\epsilon) \times B_{x_0}(\delta'_\epsilon) \times B_{u_0}(\delta'_\epsilon)$ (where $B_{x_0}(\delta')$ [resp. $B_{u_0}(\delta')$] is the ball centered in x_0 [resp. u_0] of radius δ') for some positive constants $\delta_\epsilon, \delta'_\epsilon > 0$ chosen such that $0 \le t_0 - \delta_\epsilon < t_0 + \delta_\epsilon \le T$ and $\delta'_\epsilon < \eta$.

We can construct a maximizing barrier function related to $(t_0, x_0, u_0) \in \Sigma_T^+$ with

$$\overline{\omega}_\epsilon(x) = q(t_0, x_0, u_0) + \epsilon + k_\epsilon |x - x_0|^2 + K_\epsilon \varrho(x), \tag{C.13}$$

where the parameters $K_\epsilon, k_\epsilon > 0$ are chosen large enough so that, for M_ϵ^+ the upper-bound of f on $\partial\mathcal{O}_{t_0,x_0,u_0} \cap Q_T$ (which is finite by Lemma C.3), we have

$$\overline{\omega}_\epsilon(x) - M_\epsilon^+ \ge k_\epsilon |x - x_0|^2 - M_\epsilon^+ \ge k_\epsilon (\delta'_\epsilon)^2 - M_\epsilon^+ \ge 0,$$

and, by (C.12),

$$\begin{aligned} L(\overline{\omega}_\epsilon)(x) = -2k_\epsilon u \cdot (x - x_0) - K_\epsilon u \cdot \nabla\varrho(x) &\ge -2k_\epsilon |u||x - x_0| - K_\epsilon u^{(d)} \\ &\ge -2k_\epsilon(|u_0| + \delta')\delta' + K_\epsilon \eta \ge 0. \end{aligned}$$

In the same way, we construct a minimizing barrier of the form

$$\underline{\omega}_\epsilon(t, x, u) = q(t_0, x_0, u_0) - \epsilon - \tilde{k}_\epsilon |x - x_0|^2 - \tilde{K}_\epsilon \varrho(x). \tag{C.14}$$

with $\tilde{K}_\epsilon, \tilde{k}_\epsilon > 0$ chosen so that, for M_ϵ^- the lower-bound of f on $\partial\mathcal{O}_{t_0,x_0,u_0} \cap Q_T$, we have

$$\underline{\omega}_\epsilon(x) - M_\epsilon^- \le 0 \ \text{and} \ \ L(\underline{\omega}_\epsilon)(x) \le 0.$$

Thus, $\overline{\omega}_\epsilon$ and $\underline{\omega}_\epsilon$ satisfy the properties

$$\textit{(P)-} \begin{cases} (a) \ \overline{\omega}_\epsilon(t, x, u) \ge q(t, x, u) \ge \underline{\omega}_\epsilon(t, x, u) \ \text{ for all } \ (t, x, u) \in \mathcal{O}_{t_0,x_0,u_0} \cap (0, T) \times \partial\mathcal{D} \times \mathbb{R}^d, \\ (b) \ L(\overline{\omega}_\epsilon) \ge 0 \ge L(\underline{\omega}_\epsilon) \ \text{ for all } \ (t, x, u) \in \mathcal{O}_{t_0,x_0,u_0} \cap Q_T, \\ (c) \ \overline{\omega}_\epsilon(t, x, u) \ge M^+ \ge f(t, x, u), \ \text{ and } \ \underline{\omega}_\epsilon(t, x, u) \le M^- \le f(t, x, u), \ \text{ for all } \ (t, x, u) \in \partial\mathcal{O}_{t_0,x_0,u_0} \cap Q_T, \\ (d) \ \lim\limits_{\epsilon \to 0^+} \overline{\omega}_\epsilon(t_0, x_0, u_0) = \lim\limits_{\epsilon \to 0^+} \underline{\omega}_\epsilon(t_0, x_0, u_0) = q(t_0, x_0, u_0). \end{cases}$$

Now we shall prove that, for f the solution to (6.5), $\underline{\omega}_\epsilon \le f \le \overline{\omega}_\epsilon$ on $\mathcal{O}_{t_0,x_0,u_0} \cap Q_T$. Owing to the property *(P)*-(d), this allows to conclude that $f(t, x, u)$ tends to $q(t_0, x_0, u_0)$ as (t, x, u) tends to (t_0, x_0, u_0), for all (t_0, x_0, u_0) of Σ_T^+.

For the local comparison between $\overline{\omega}_\epsilon$ and f, we proceed as in the proof of Lemma C.3 and we consider the positive part $(f - \overline{\omega}_\epsilon)^+$ of $f - \overline{\omega}_\epsilon$. Let β be a real parameter that we will specify later. Recalling from the proof of Lemma C.3 that the function $\triangle_u |(f - \overline{\omega}_\epsilon)^+|^2$ is well defined a.e. on Q_T with

$$\triangle_u |(f - \overline{\omega}_\epsilon)^+|^2 = 2((f - \overline{\omega}_\epsilon)^+ \triangle_u (f - \overline{\omega}_\epsilon)) + 2\left|\nabla_u (f - \overline{\omega}_\epsilon)\right|^2 \mathbb{1}_{\{f > \overline{\omega}_\epsilon\}}.$$

we shall observe that, on $\mathcal{O}_{t_0,x_0,u_0} \cap Q_T$,

$$L(\exp\{\beta t\} \left|(f - \overline{\omega}_\epsilon)^+\right|^2) = \beta \exp\{\beta t\} \left|(f - \overline{\omega}_\epsilon)^+\right|^2 + \exp\{\beta t\} L(\left|(f - \overline{\omega}_\epsilon)^+\right|^2).$$

The property $(P-)(b)$ ensures that

$$L(\left|(f-\overline{\omega}_\epsilon)^+\right|^2) = 2(f-\overline{\omega}_\epsilon)^+L(f-\overline{\omega}_\epsilon) - \sigma^2\left|\nabla_u(f-\overline{\omega}_\epsilon)\right|^2 \mathbb{1}_{\{f>\overline{\omega}_\epsilon\}} \leq -\sigma^2\left|\nabla_u(f-\overline{\omega}_\epsilon)\right|^2 \mathbb{1}_{\{f>\overline{\omega}_\epsilon\}} \leq 0,$$

so that

$$L(\exp\{\beta t\}\left|(f-\overline{\omega}_\epsilon)^+\right|^2) \leq \beta\exp\{\beta t\}\left|(f-\overline{\omega}_\epsilon)^+\right|^2.$$

Integrating the two sides of the above inequality over $\mathcal{O}_{t_0,x_0,u_0}\cap Q_T$, we get

$$\int_{\mathcal{O}_{t_0,x_0,u_0}\cap Q_T} L(\exp\{\beta t\}\left|(f-\overline{\omega}_\epsilon)^+\right|^2) \leq \int_{\mathcal{O}_{t_0,x_0,u_0}\cap Q_T} \beta\exp\{\beta t\}\left|(f-\overline{\omega}_\epsilon)^+\right|^2.$$

wing to $(P)-(a)$ and $(P)-(c)$, $(f-\overline{\omega}_\epsilon)^+$ is zero on $\mathcal{O}^\epsilon_{t_0,x_0,u_0}\cap\Sigma_T$ and $\partial\mathcal{O}^\epsilon_{t_0,x_0,u_0}\cap Q_T$. An integration by parts of the left-hand side expression yields

$$\begin{aligned}&\int_{\mathcal{O}^\epsilon_{t_0,x_0,u_0}\cap Q_T} L(\exp\{\beta t\}\left|(f-\overline{\omega}_\epsilon)^+\right|^2)(t,x,u)\\ &= \int_{\mathcal{O}^\epsilon_{t_0,x_0,u_0}\cap Q_T} \exp(\beta t)\left|(f-\overline{\omega}_\epsilon)^+\right|^2(t,x,u)\nabla_u\cdot b(x,u),\end{aligned}$$

and

$$0 \leq \int_{\mathcal{O}_{t_0,x_0,u_0}\cap Q_T} (\beta - \nabla_u\cdot b(x,u))\exp\{\beta t\}|(f-\overline{\omega}_\epsilon)^+|^2.$$

Choosing $\beta < 0$ such that $\beta + \|\nabla_u\cdot b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d)} < 0$ ensures that, for a.e. $(t,x,u)\in\mathcal{O}^\epsilon_{t_0,x_0,u_0}\cap Q_T$, $f(t,x,u)\leq\overline{\omega}_\epsilon(t,x,u)$. Similar arguments entail that $\underline{\omega}_\epsilon\leq f$. □

Feynman-Kac representation and continuity up to and along Σ^-_T. We prove the Feynman-Kac representation (6.6) by replicating the arguments of Friedman [14, Chapter 5, Theorem 5.2]: for $(y,v)\in\mathcal{D}\times\mathbb{R}^d$ fixed, let $((x^{y,v}_t,u^{y,v}_t);\ t\in[0,T])$ satisfy (6.4). Set $\beta^{y,v}_\delta := \inf\{t>0\,;\ d(x^{y,v}_t,\partial\mathcal{D})\leq\delta\}$. Since f is smooth in the interior of Q_T and satisfies (6.5), applying Itô's formula to $f(t-s,x^{y,v}_{s\wedge\beta^{y,v}_\delta},u^{y,v}_{s\wedge\beta^{y,v}_\delta})$, for $s\in[0,t]$, yields

$$f(t,y,v) = \mathbb{E}_\mathbb{P}\left[f_0(x^{y,v}_t,u^{y,v}_t)\mathbb{1}_{\{t\,\leq\,\beta^{y,v}_\delta\}}\right] + \mathbb{E}_\mathbb{P}\left[f(t-\beta^{y,v}_\delta,x^{y,v}_{\beta^{y,v}_\delta},u^{y,v}_{\beta^{y,v}_\delta})\mathbb{1}_{\{t\,>\,\beta^{y,v}_\delta\}}\right].$$

Since $\mathbb{P}$-a.s., $\beta^{y,v}_\delta$ tends to $\beta^{y,v} = \inf\{t>0\,;\ d(x^{y,v}_t,\partial\mathcal{D})=0\}$, as δ tends to 0, and thanks to Proposition C.4, one obtains (6.6).

Proposition C.5. *Assume $(H_{f_0,q})$. Let $f\in\mathcal{C}^{1,1,2}(Q_T)\cap\mathcal{C}(Q_T\cup\Sigma^+_T)$ be the solution to (6.5). Then f is continuous along and up to Σ^-_T.*

Proof. According to (6.6) and since f_0 and q are continuous, the continuity of f up to Σ^-_T will follow from the continuity of $(y,v)\mapsto(\beta^{y,v},x^{y,v}_t,u^{y,v}_t)$. $\mathbb{P}$-almost surely, for all $t\geq0$, the flow $(y,v)\mapsto(x^{y,v}_t,u^{y,v}_t)$ is continuous on $\mathbb{R}^d\times\mathbb{R}^d$. As $(y,v)\notin\Sigma^0\cup\Sigma^+$, we have $\beta^{y,v}=\tau^{y,v} := \inf\{t>0;\ x^{y,v}_t\notin\overline{\mathcal{D}}\}$. To prove that $(y,v)\mapsto\tau^{y,v}$ is continuous up to Σ^-, we follow the general proof of the continuity of exit time related to a flow of continuous processes given in Proposition 6.3 in Darling and Pardoux [13]. First, replicating the argument of the authors, one can show that, for all $(y_m,v_m)\in\mathcal{D}\times\mathbb{R}^d$ such that $\lim_{m\to+\infty}(y_m,v_m)=(y,v)\in\Sigma^-$,

$$\limsup_{m\to+\infty}\tau^{y_m,v_m}\leq\tau^{y,v}.$$

Next, it is sufficient to check that

$$\tau^{y,v}\leq\liminf_{m\to+\infty}\tau^{y_m,v_m}.$$

By an [6] it is shown that for a.e. $(y,v)\in\mathcal{D}\times\mathbb{R}^d\cup\Sigma^-$, the path $t\mapsto(x^{y,v}_t,u^{y,v}_t)$ never hits $\Sigma^0\cup\Sigma^-$, and, since $\mathbb{P}$-a.s. $(t,y,v)\mapsto(x^{y,v}_t,u^{y,v}_t)$ is continuous on $[0,+\infty)\times\overline{\mathcal{D}}\times\mathbb{R}^d$, one can check that

$$\overline{\{(x^{y_m,v_m}_{\tau^{y_m,v_m}},u^{y_m,v_m}_{\tau^{y_m,v_m}});\ m\in\mathbb{N}\}}\subset\Sigma^+,$$

and that $(x^{y,v}_{\liminf_{m\to+\infty}\tau^{y_m,v_m}},u^{y,v}_{\liminf_{m\to+\infty}\tau^{y_m,v_m}})\in\Sigma^+$. Since $\tau^{y,v}=\inf\{t>0;\ (x^{y,v}_t,u^{y,v}_t)\in\Sigma^+\}$, we deduce that $\tau^{y,v}\in[0,\liminf_{m\to+\infty}\tau^{y_m,v_m}]$. □

C.2 Proof of Corollary 6.2

Proof. For $n > 1$, let us assume that $\Gamma^{\psi}_{n-1} \in \mathcal{C}(\overline{Q_T} \setminus \Sigma^0)$ with $\Gamma^{\psi}_{n-1}|_{\Sigma^-_T} \in L^2(\Sigma^-_T)$. Then $\Gamma^{\psi}_{n-1}(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))|_{\Sigma^+_T}$ is in $L^2(\Sigma^+_T)$ since, by using the change of variables

$$u \mapsto \widehat{u} := u - 2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)$$

for fixed $x \in \partial\mathcal{D}$, we have

$$\begin{aligned}&\int_{\Sigma^+_T} |(u\cdot n_{\mathcal{D}}(x))| \left(\Gamma^{\psi}_{n-1}(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))\right)^2 d\lambda_{\Sigma_T}(t,x,u)\\&= \int_{\Sigma^-_T} |(u\cdot n_{\mathcal{D}}(x))| \left(\Gamma^{\psi}_{n-1}(t,x,u)\right)^2 d\lambda_{\Sigma_T}(t,x,u) = \|\Gamma^{\psi}_{n-1}\|^2_{L^2(\Sigma^-_T)} < +\infty.\end{aligned} \tag{C.15}$$

From the strong Markov property of the solution of (1.1), we get that for all $(x,u) \in (\mathcal{D}\times\mathbb{R}^d)\cup\Sigma^+$,

$$\mathbb{E}[\psi(X^{x,u}_{t\wedge\tau^{x,u}_n}, U^{x,u}_{t\wedge\tau^{x,u}_n})\mathbb{1}_{\{\tau^{x,u}_1<t\}}] = \mathbb{E}[\Gamma^{\psi}_{n-1}(t-\tau^{x,u}_1, X^{x,u}_{\tau^{x,u}_1}, U^{x,u}_{\tau^{x,u}_1})\mathbb{1}_{\{\tau^{x,u}_1<t\}}]. \tag{C.16}$$

Considering a sequence $(x_m,u_m, m\in\mathbb{N})$ in $\mathcal{D}\times\mathbb{R}^d$ converging to $(x,u)\in\Sigma^+$, and $t>0$, we have, for m large enough

$$\begin{aligned}(\tau^{x_m,u_m}_1, X^{x_m,u_m}_t, U^{x_m,u_m}_t, \{t<\tau^{x_m,u_m}_1\}) &= (\beta^{x_m,u_m}, x^{x_m,u_m}_t, u^{x_m,u_m}_t, \{t<\beta^{x_m,u_m}\})\\ (X^{x_m,u_m}_{\tau^{x_m,u_m}_1}, U^{x_m,u_m}_{\tau^{x_m,u_m}_1}) &= (x^{x_m,u_m}_{\beta^{x_m,u_m}}, \widehat{u}^{x_m,u_m}_{\beta^{x_m,u_m}}).\end{aligned}$$

Hence, from the continuity of $(y,v)\mapsto(\beta^{y,v}, x^{y,v}_t, u^{y,v}_t)$ proved with Proposition C.5,

$$\lim_{m\to+\infty}(\tau^{x_m,u_m}_1, X^{x_m,u_m}_{t\wedge\tau^{x_m,u_m}_1}, U^{x_m,u_m}_{t\wedge\tau^{x_m,u_m}_1}) = (0,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)).$$

Since Ψ is continuous and $\Psi = 0$ on Σ^+, the right-hand side of (C.16) is then continuous on $(\mathcal{D}\times\mathbb{R}^d)\cup\Sigma^+$, as well as

$$\mathbb{E}[\psi(X^{x,u}_{t\wedge\tau^{x,u}_n}, U^{x,u}_{t\wedge\tau^{x,u}_n})\mathbb{1}_{\{\tau^{x,u}_1\geq t\}}] = \mathbb{E}[\psi(X^{x,u}_{t\wedge\tau^{x,u}_1}, U^{x,u}_{t\wedge\tau^{x,u}_1})\mathbb{1}_{\{\tau^{x,u}_1\geq t\}}].$$

Moreover, for $(t,x,u)\in\Sigma^+_T$,

$$\begin{aligned}\Gamma^{\psi}_n(t,x,u) &= \lim_{m\to+\infty}\left\{\mathbb{E}[\psi(X^{x_m,u_m}_{t\wedge\tau^{x_m,u_m}_n}, U^{x_m,u_m}_{t\wedge\tau^{x_m,u_m}_n})\mathbb{1}_{\{\tau^{x_m,u_m}_1<t\}}] + \mathbb{E}[\psi(X^{x_m,u_m}_{t\wedge\tau^{x_m,u_m}_n}, U^{x_m,u_m}_{t\wedge\tau^{x_m,u_m}_n})\mathbb{1}_{\{\tau^{x_m,u_m}_1\geq t\}}]\right\}\\&= \Gamma^{\psi}_{n-1}(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)).\end{aligned}$$

Now Theorem 6.1 ensures that there exists a classical solution f_n to (6.5) for $f_0 = \psi$ and $q(t,x,u) = \Gamma^{\psi}_{n-1}(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))$ on Σ^+_T. According to (6.6), we have, for $(t,x,u)\in Q_T$

$$\begin{aligned}f_n(t,x,u) &= \mathbb{E}_{\mathbb{P}}\left[\psi(x^{x,u}_t, u^{x,u}_t)\mathbb{1}_{\{t\,\leq\,\beta^{x,u}\}}\right]\\&\quad+\mathbb{E}_{\mathbb{P}}\left[\Gamma^{\psi}_{n-1}\left(t-\beta^{x,u}, x^{x,u}_{\beta^{x,u}}, u^{x,u}_{\beta^{x,u}} - 2(u^{x,u}_{\beta^{x,u}}\cdot n_{\mathcal{D}}(x^{x,u}_{\beta^{x,u}}))n_{\mathcal{D}}(x^{x,u}_{\beta^{x,u}})\right)\mathbb{1}_{\{t\,>\,\beta^{x,u}\}}\right].\end{aligned}$$

One can observe that

$$\mathbb{E}_{\mathbb{P}}\left[\psi(x^{x,u}_t, u^{x,u}_t)\mathbb{1}_{\{t\,\leq\,\beta^{x,u}\}}\right] = \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_t, U^{x,u}_t)\mathbb{1}_{\{t\,\leq\,\tau^{x,u}_1\}}\right] = \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_{t\wedge\tau^{x,u}_n}, U^{x,u}_{t\wedge\tau^{x,u}_n})\mathbb{1}_{\{t\,\leq\,\tau^{x,u}_1\}}\right]$$

and that

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\left[\Gamma^{\psi}_{n-1}\left(t-\beta^{x,u}, x^{x,u}_{\beta^{x,u}}, u^{x,u}_{\beta^{x,u}} - 2(u^{x,u}_{\beta^{x,u}}\cdot n_{\mathcal{D}}(x^{x,u}_{\beta^{x,u}}))n_{\mathcal{D}}(x^{x,u}_{\beta^{x,u}})\right)\mathbb{1}_{\{t\,>\,\beta^{x,u}\}}\right]\\&= \mathbb{E}_{\mathbb{P}}\left[\Gamma^{\psi}_{n-1}(t-\tau^{x,u}_1, X^{x,u}_{\tau^{x,u}_1}, U^{x,u}_{\tau^{x,u}_1})\mathbb{1}_{\{t\,>\,\tau^{x,u}_1\}}\right]\\&= \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_{t\wedge\tau^{x,u}_n}, U^{x,u}_{t\wedge\tau^{x,u}_n})\mathbb{1}_{\{t\,>\,\tau^{x,u}_1\}}\right],\end{aligned}$$

where the second equality follows from the strong Markov property of $(X_t^{x,u}, U_t^{x,u})$. Therefore

$$f_n(t,x,u) = \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_{t\wedge\tau_n^{x,u}}, U^{x,u}_{t\wedge\tau_n^{x,u}})\right] = \Gamma_n^\psi(t,x,u),$$

from which we deduce that $\Gamma_n^\psi \in \mathcal{C}_b^{1,1,2}(Q_T)\cap\mathcal{C}(\overline{Q_T}\setminus\Sigma_T^0)\cap L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ is a solution to (6.8) with $\Gamma_n^\psi|_{\Sigma_T^-}\in L^2(\Sigma_T^-)$. Moreover, according to (C.15), for all $t\in(0,T)$,

$$\begin{aligned}&\|\Gamma_n^\psi(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\Gamma_n^\psi\|^2_{L^2(\Sigma_t^-)} + \int_{Q_t}(\nabla_u\cdot b(x,u))(\Gamma_n^\psi)^2 + \sigma^2\|\nabla_u\Gamma_n^\psi\|^2_{L^2(Q_t)}\\ &= \|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)} = \|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\Gamma_{n-1}^\psi\|^2_{L^2(\Sigma_t^-)}.\end{aligned} \tag{C.17}$$

which implies

$$\begin{aligned}&\|\Gamma_n^\psi(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\Gamma_n^\psi\|^2_{L^2(\Sigma_t^-)} + \sigma^2\|\nabla_u\Gamma_n^\psi\|^2_{L^2(Q_t)}\\ &\le \|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\Gamma_{n-1}^\psi\|^2_{L^2(\Sigma_t^-)} + \|\nabla_u\cdot b\|_{L_\infty(\mathcal{D}\times\mathbb{R}^d)}\int_{Q_t}(\Gamma_n^\psi)^2.\end{aligned}$$

resulting in (6.9) by Gronwall's lemma. For $n=1$, setting $f_0=\psi$ and $q=\psi|_{\Sigma_T^+}=0$ (since ψ has its support in the interior of $\mathcal{D}\times\mathbb{R}^d$), one can check that $\Gamma_1^\psi\in\mathcal{C}_b^{1,1,2}(Q_T)\cap\mathcal{C}(\overline{Q_T}\setminus\Sigma_T^0)\cap L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ satisfies (6.8) and (6.9). By induction, we end the proof. □

C.3 Proof of Corollary 6.3

Proof. We first observe that since $\psi|_{\partial\mathcal{D}\times\mathbb{R}^d}=0$,

$$\Gamma_n^\psi(t,x,u) = \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_{t\wedge\tau_n^{x,u}}, U^{x,u}_{t\wedge\tau_n^{x,u}})\right] = \mathbb{E}_{\mathbb{P}}\left[\psi(X_t^{x,u},U_t^{x,u})\mathbb{1}_{\{\tau_n^{x,u}\ge t\}}\right].$$

Next, there exists a nonnegative function $\beta\in L^2(\mathbb{R}\times\mathbb{R})$ such that $\beta(|x|,|u|)=1$ on the support of ψ and $|\psi|\le C\beta(|x|,|u|)$, with $C:=\sup_{(x,u)\in\mathcal{D}\times\mathbb{R}^d}|\psi|(x,u)$. Then

$$\Gamma_n^\psi(t,x,u)\le C\mathbb{E}_{\mathbb{P}}\left[\beta(|X_t^{x,u}|,|U_t^{x,u}|)\mathbb{1}_{\{\tau_n^{x,u}\ge t\}}\right].$$

As $\mathbb{E}_{\mathbb{P}}[\beta(|U_t^{x,u}|)]$ is equal to the convolution product $(G*\beta)(|x|,|u|)$, where G denotes the density of the free Langevin process (6.4), we obtain

$$-C(G*\beta)(|x|,|u|)\le\Gamma_n^\psi(t,x,u)\le C(G*\beta)(|x|,|u|),\ \text{on } Q_T. \tag{C.18}$$

Owing to the continuity of Γ_n^ψ, from the interior of Q_T to its boundary, (C.18) still holds true along $\Sigma_T^\pm$.

It is show in **Proposition 3.1** from [5] that for a.e. $(x,u)\in(\mathcal{D}\times\mathbb{R}^d)\cup(\Sigma\setminus\Sigma^0)$, $\mathbb{P}_{(x,u)}$-a.s. τ_n grows to ∞ as n increases, so then

$$\lim_{n\to+\infty}\Gamma_n^\psi(t,x,u)=\Gamma^\psi(t,x,u),\ \text{for a.e. } (t,x,u)\in Q_T,\ \lambda_{\Sigma_T}\text{-a.e. } (t,x,u)\in\Sigma_T\setminus\Sigma_T^0. \tag{C.19}$$

Indeed,

$$\left|\Gamma_n^\psi(t,x,u)-\Gamma^\psi(t,x,u)\right| = \left|\mathbb{E}_{\mathbb{P}}\left[\psi(X_t^{x,u},U_t^{x,u})\mathbb{1}_{\{\tau_n^{x,u}\le t\}}\right]\right|\le\|\psi\|_\infty\mathbb{P}(\tau_n^{x,u}\le t).$$

In particular, (C.18) is also true for $\Gamma^\psi(t)$. We conclude by the Lebesgue Dominated Convergence Theorem that $\Gamma_n^\psi(t)$ converges to $\Gamma^\psi(t)$ in $L^2(\mathcal{D}\times\mathbb{R}^d)$. And since Γ_n^ψ is continuous on the compact $[0,T]$ we have the convergence to Γ^ψ in $L^2(Q_T)$. The Lebesgue Dominated Convergence Theorem also shows that:

$$\int_{Q_t}(\nabla_u\cdot b(x,u))(\Gamma_n^\psi)^2\to\int_{Q_t}(\nabla_u\cdot b(x,u))(\Gamma^\psi)^2.$$

Next we deduce that the norms involving Γ_n^ψ in the left-hand side of (C.17) are finite for all t, uniformly in n (as the right-hand side of (6.9) is bounded uniformly in n by the Maxwellian bound (C.18) and ψ is of compact support). Therefore, the estimate (C.17) is also true for Γ^ψ (see e.g. [9]), and Γ_n^ψ converges to Γ^ψ in $L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ and the equality (C.17) becomes:

$$\|\Gamma^\psi(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}+\int_{Q_t}(\nabla_u\cdot b(x,u))(\Gamma^\psi)^2+\sigma^2\|\nabla_u\Gamma^\psi\|^2_{L^2(Q_t)}=\|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} \tag{C.20}$$

and by Gronwall's lemma, we obtain (6.10). □

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